Lipschitz Embeddings into $\mathbb{R}^n$

by

PATRICE ASSOUAD (*)

Translated by Junhyung Park[1]

ABSTRACT. – A Lipschitz embedding of a metric space (X, d) into another one (Y, δ) is a map $f : X \to Y$ such that

$$\exists A, B \in]0, +\infty[, \quad \forall x, x' \in X, \quad Ad(x, x') \leqslant \delta(f(x), f(x')) \leqslant Bd(x, x').$$

We describe here three methods to obtain Lipschitz embeddings of the metric space $(\mathbb{R}^k, \| \ \|^p)$ into some metric space $(\mathbb{R}^n, \| \ \|)$.

The third method allows us to minimise, for $k = 1$, the rank of such an embedding (i.e. to obtain the minimal value of the integer n).

Introduction

Let (X, d) and (Y, δ) be two metric spaces. A map $f : X \to Y$ is called a *Lipschitz embedding of* (X, d) *into* (Y, δ) if there exist two numbers $A, B \in]0, +\infty[$ such that we have

$$Ad(x, x') \leqslant \delta(f(x), f(x')) \leqslant Bd(x, x')$$

for any $x, x' \in X$ (if we want to be more precise, we say that f is an (A, B)-*Lipschitz embedding of* (X, d) *into* (Y, δ)).

In this regard, several questions seem interesting to me:

– how to recognise if there exists, for a given metric space (X, d), an integer n and a Lipschitz embedding of (X, d) into $(\mathbb{R}^n, \| \ \|)$?

– if it is the case, to calculate the rank of (X, d) (i.e. to minimise n);

– to study how, as n increases to $+\infty$, the distortion of (X, d) in $(\mathbb{R}^n, \| \ \|)$ decreases (that is to say, the lower bound of $\log\left(\frac{B}{A}\right)$ of all Lipschitz embeddings of (X, d) into $(\mathbb{R}^n, \| \ \|)$).

In the present work, we will study in particular the metric space $(\mathbb{R}^k, \| \ \|^p)$ from this point of view:

(a) we will describe three methods of embedding $(\mathbb{R}^k, \| \ \|^p)$ into $(\mathbb{R}^n, \| \ \|)$:

– the first method (very simple, but which does not give a good evaluation of the rank nor the distortion) can, in particular, be carried out using lacunary series or Schauder basis;

*Text received on 17 May 1983. P. ASSOUAD, Université de Paris Sud, Centre d'Orsay, Département de Mathématique, Bâtiment 425, 91405 Orsay, Cedex.

[1]This is not authoritative translation, and I do not claim any credit for the mathematical content of this document. Please send any corrections to `junhyung.park@tuebingen.mpg.de`.

– the second method, which generalises the preceding one, allows the embedding of the space (X, d^p) into a space $(\mathbb{R}^n, \| \ \|)$, for any number $p \in]0,1[$, for every metric space (X, d) of finite metric dimension (these spaces are studied notably in [1]);

– the third method uses generalised KOCH curves (in fact this method extends the observation, due to GLAESER [5] p.57, that the classical curve of H. VON KOCH [7] realises a Lipschitz embedding of the space $([0,1], \| \ \|^{\mathrm{Log}3/\mathrm{Log}4})$ into $(\mathbb{R}^2, \| \ \|)$).

(*b*) We show that this third method allows us, at least for $k = 1$, to obtain the exact value of the rank of the space $([0,1]^k, \| \ \|^p)$.

So here is how this work is divided:

Section 1 SCHAUDER basis and lacunary series.

Section 2 Metric dimension and embedding.

Section 3 Generalised KOCH curves.

Section 4 Rank of the space $([0,1]^k, \| \ \|^p)$.

Acknowledgements

This work was first written in English (March 1981); I would like to express my gratitude towards those who encouraged me to write it in the present form.

Notations

– Let $k \geqslant 1$ be an integer; we always denote by $\| \ \|$ the Euclidean norm on $\mathbb{R}^k$.

– Let $p \in]0,1[$; if T is a subset of the Euclidean space $(E, \| \ \|)$, we denote by $(T, \| \ \|^p)$ the set T equipped with the distance $s, t \to \|s - t\|^p$

(this is really to do with a metric space, not a normed space).

– Let $b \in]0, +\infty[$; a subset T of a metric space (X, d) is said to be *b-separated* (resp. of diameter $\leqslant b$) if we have $d(s,t) > b$ (resp. $d(s,t) \leqslant b$) for every pair (s, t) of distinct points of T.

– Let x and y be two real numbers; we then write:

$x \wedge y$ for $\mathrm{Inf}(x, y)$, $\quad x \vee y$ for $\mathrm{Sup}(x, y)$, $\quad$ and x^+ for $\mathrm{Sup}(x, 0)$.

1 Schauder basis and lacunary series

We are first going to describe *quite a general construction* (it can be carried out using Schauder basis or lacunary series, as we will see in the following; it can equally be applied to Section 2):

1.1. (*a*) Let φ be a map from a space (X, d) into a Hilbert space $(E, \| \ \|)$, and let $\tau \in]0,1[$ and $A, B, c \in]0, +\infty[$; we say that φ is a *local* (τ, A, B, c)*-controlled embedding of* (X, d) *into* $(E, \| \ \|)$ if it satisfies the following conditions for all $x, t \in X$:

(*a*1) $\tau c < d(s,t) \leqslant c$ implies $\|\varphi(s) - \varphi(t)\| \geqslant A$;

(*a*2) $\|\varphi(s) - \varphi(t)\| \leqslant B(d(s,t) \wedge 1)$;

(*b*) let $(v_j)_{j \in \mathbb{Z}}$ be a sequence, periodic with period $2N$, of elements of a Euclidean space; we say that $(v_j)_{j \in \mathbb{Z}}$ is a *cyclic base of* $(F, \| \ \|)$ if $(v_1, ..., v_{2N})$ is an orthonormal basis of $(F, \| \ \|)$.

1.2. Proposition. – *Let* $\tau, p \in]0,1[$ *and* $A, B, c \in]0,+\infty[$. *Let* (X,d) *be a metric space with a designated 0 element, and let* $(E, \| \ \|)$ *be a Hilbert space.*

For every $j \in \mathbb{Z}$, *take a local* (τ, A, B, c)*-controlled embedding* φ_j *such that* $\varphi_j(0) = 0$, *of the space* $(X, \tau^{-j}d)$ *into* $(E, \| \ \|)$. *Also take a cyclic basis* $(v_j)_{j\in\mathbb{Z}}$ *of a Euclidean space* $(F, \| \ \|)$ *of dimension* $2N$.

For every $s \in X$, *we let* $f(s) = \sum_{j\in\mathbb{Z}} \tau^{jp}\varphi_j(s) \otimes v_j$. *Then we have*

$$B\left(\frac{c^{-p}}{1-\tau^p} + \frac{c^{1-p}}{1-\tau^{1-p}}\right) d^p(s,t) \geqslant \|f(s) - f(t)\|$$

$$\geqslant \left[Ac^{-p} - B\left(\frac{c^{-p}}{1-\tau^p}\tau^{Np} + \frac{c^{1-p}}{1-\tau^{1-p}}\tau^{N(1-p)}\right)\right] d^p(s,t).$$

In particular, the map f *is a Lipschitz embedding of* (X, d^p) *into* $(E \otimes F, \| \ \|)$, *as soon as* N *is sufficiently large to have* $\frac{A}{B} > \frac{\tau^{Np}}{1-\tau^p} + c\frac{\tau^{N(1-p)}}{1-\tau^{1-p}}$ *(*$E \otimes F$ *is equipped with the tensor product of the Hilbert structures of* E *and* F*).*

Proof. – For each $j \in \mathbb{Z}$, we denote by f_j the map $s \to \tau^{jp}\varphi_j(s) \otimes v_j$. We have $f(0) = 0$; for the upper bound of $\|f(s) - f(t)\|$ that we propose to establish, it will thus suffice to show the convergence of the series $f = \sum_{j\in\mathbb{Z}} f_j$. Let (s,t) be a pair of distinct points of X, and let l be a natural number such that $\tau^{l+1} < \frac{1}{c}d(s,t) \leqslant \tau^l$.

(*a*) *(Upper bound).* We then have:

$$\begin{aligned}
\|f(s) - f(t)\| &\leqslant \sum_{j\geqslant l+1} \|f_j(s) - f_j(t)\| + \sum_{j\leqslant l} \|f_j(s) - f_j(t)\| \\
&\leqslant \sum_{j\geqslant l+1} B\tau^{jp} + \sum_{j\leqslant l} B\tau^{j(p-1)}d(s,t) \\
&\leqslant B\left(\frac{1}{1-\tau^p}\tau^{(l+1)p} + \frac{d(s,t)}{1-\tau^{1-p}}\tau^{l(p-1)}\right) \\
&\leqslant B\left(\frac{c^{-p}}{1-\tau^p} + \frac{c^{1-p}}{1-\tau^{1-p}}\right) d^p(s,t).
\end{aligned}$$

(*b*) *(Lower bound).* We also have:

$$\begin{aligned}
\|f(s) - f(t)\| \geqslant &\left\| \sum_{-N<j-l\leqslant N} (f_j(s) - f_j(t)) \right\| \\
&- \sum_{j-l>N} \|f_j(s) - f_j(t)\| - \sum_{j-l\leqslant -N} \|f_j(s) - f_j(t)\| \\
\geqslant &\|f_l(s) - f_l(t)\| - \sum_{j-l>N} B\tau^{jp} + \sum_{j-l\leqslant -N} B\tau^{j(p-1)}d(s,t) \\
\geqslant &A\tau^{lp} - B\left(\frac{1}{1-\tau^p}\tau^{(N+l+1)p} + \frac{d(s,t)}{1-\tau^{1-p}}\tau^{(l-N)(p-1)}\right) \\
\geqslant &\left[Ac^{-p} - B\left(\frac{c^{-p}}{1-\tau^p}\tau^{Np} + \frac{c^{1-p}}{1-\tau^{1-p}}\tau^{N(1-p)}\right)\right] d^p(s,t).
\end{aligned}$$

□

Let us expound upon the preceding Proposition a little:

1.3. (a) If E is of dimension M, we have obtained a Lipschitz embedding of the space (X, d^p) into $(\mathbb{R}^{2NM}, \| \ \|)$; but $2NM$ is not a good evaluation of the rank of the space (X, d^p), as we will see in Section 4.

(b) We can note that the above construction does not work well as p tends to 1 (even when the embedding is trivial for $p = 1$).

(c) Let $m \in \mathbb{Z}$; if T is a subset of diameter $\leqslant c\tau^m$ of the space (X, d), then the map $s \to \sum_{j \geqslant m} \tau^{jp} \varphi_j(s) \otimes v_j$ is a Lipschitz embedding of the space (T, d^p) into $(E \otimes F, \| \ \|)$, as long as N is sufficiently large.

(d) Let $m \in \mathbb{Z}$; if T is a $c\tau^m$-separated subset of the space (X, d), then the map $s \to \sum_{j < m} \tau^{jp} \varphi_j(s) \otimes v_j$ is a Lipschitz embedding of the space (T, d^p) into $(E \otimes F, \| \ \|)$, as long as N is sufficiently large ((c) and (d) can be shown exactly as Proposition 1.2, but taking into account the values that l can take).

1.4. Corollary. – *Let* $\tau, p \in]0, 1[$ *and* $A, B, c \in]0, +\infty[$. *Let* φ *be a local* (τ, A, B, c)*-controlled embedding of* $\mathbb{R}^k$ *into a Euclidean space* $(E, \| \ \|)$ *and let* $(v_j)_{j \in \mathbb{Z}}$ *be a cyclic base of a Euclidean space* $(E, \| \ \|)$ *of dimension* $2N$.

For each $s \in \mathbb{R}^k$, *we let* $f(s) = \sum_{j \in \mathbb{Z}} \tau^{jp} \varphi(\tau^{-j} s) \otimes v_j$. *Then* f *is a Lipschitz embedding of* $(\mathbb{R}^k, \| \ \|^p)$ *into* $(E \otimes F, \| \ \|)$, *as long as* N *is sufficiently large.*

Proof. – Indeed, for each $j \in \mathbb{Z}$, the map $\varphi_j : s \to \varphi(\tau^{-j} s)$ is a local (τ, A, B, c)-controlled embedding of the space $(\mathbb{R}^k, \tau^{-j} \| \ \|)$ into $(E, \| \ \|)$. □

Let us describe some *examples of local embeddings of* $\mathbb{R}$ *into a Euclidean space*:

(1.5) Let us fix $\tau \in]0, 1[$ and take $E = \mathbb{C}$ (considered as a Euclidean space of dimension 2 on $\mathbb{R}$). The function $\varphi : s \to \exp(is) - 1$ is then a local $(\tau, A, 2, c)$-controlled embedding of $\mathbb{R}$ in $\mathbb{C}$ (by taking, for example, $c = \frac{2\pi}{1+\tau}$ and $A = |\varphi(c\tau)|$).

In particular, we can then embed $([0, 1], \| \ \|^p)$ into a Euclidean space using a lacunary Fourier series with appropriate vectorial coefficients.

(1.6) Let us fix $\tau \in]0, 1[$ and take $E = \mathbb{R}^2 \otimes \mathbb{R}^2$ (the canonical base of $\mathbb{R}^2$ will be denoted by (e_1, e_2)). For each $j \in \mathbb{Z}$ and each $z \in D_j = 2^{-j}\mathbb{Z}$, we denote by $\Delta_{j,z}$ the functions $s \to (1 - |1 + 2^{j+1}(z - s)|)^+$ (*Schauder basis*). Let $\lambda \in \mathbb{R} \backslash \{0\}$; the function

$$\varphi_\lambda = \sum_{r \text{ even}} e_1 \otimes (e_1 \Delta_{0,r} + \lambda e_2 \Delta_{1,r/2}) + \sum_{r \text{ odd}} e_2 \otimes (e_1 \Delta_{0,r} + \lambda e_2 \Delta_{1,r/2})$$

is then a local (τ, A, B, c)-controlled embedding of $\mathbb{R}$ into E, for appropriate values of A, B and c (because φ_λ is injective, bounded, Lipschitz and periodic on every interval of width < 1).

1.7. Corollary. – *In particular, let us take* $\tau = \frac{1}{4}$ *and* $\lambda = 2^{-p}$. *The embedding* f *of* $(\mathbb{R}, \| \ \|^p)$ *into a Euclidean space constructed using the local embedding* φ_λ *can be written as* $f = \sum_{j \in \mathbb{Z}} \sum_{z \in D_j} 2^{-jp} u'_{j,z} \otimes v'_j \Delta_{j,z}$, *where the sequence* $(v'_j)_{j \in \mathbb{Z}}$ *(resp. for each* $j \in \mathbb{Z}$, *the sequence* $r \to u'_{j, r2^{-j}}$*) is a cyclic basis of a Euclidean space* F' *(resp.* E'*) of appropriate dimension.*

Proof. – For each $j \in \mathbb{Z}$, we let $v'_{2j} = e_1 \otimes v_j$ and $v'_{2j+1} = e_2 \otimes v_j$; moreover, for each $z \in D_j$, we let $u'_{j,z} = e_1$ if $z2^j$ is even, and $u'_{j,z} = e_2$ otherwise. The sequence $(v'_j)_{j\in\mathbb{Z}}$ is thus a cyclic basis of $F' = \mathbb{R}^2 \otimes F$; moreover, for each $j \in \mathbb{Z}$, the sequence $r \to u'_{j,r2^{-j}}$ is a cyclic basis of $E' = \mathbb{R}^2$.

We immediately have that the embedding f is of the form announced in the statement. □

We will see in the next Section that the embedding process described in (1.2) can be applied to quite general metric spaces.

2 Metric dimension and embedding

2.1. we say that a metric space (X, d) is (C, s)-*homogeneous* if we have $|Y \cap Z| \leqslant C \left(\frac{b}{a}\right)^s$ for any $a, b \in]0, +\infty[$ $(a < b)$ and for any a-separated Y (resp. Z of diameter $\leqslant b$) in the space (X, d). The *metric dimension of* (X, d) (which is denoted as $\mathrm{Dim}(X, d)$) is the infimum of real numbers $s \geqslant 0$ such that, for a certain $C \in]0, +\infty[$, the space (X, d) is (C, s)-homogeneous.

This notion of metric dimension (there are others) is old, since it goes back to an article of G. Bouligand [3] (1928). The results which will follow are more recent, and appear in my thesis [1] (1977).

2.2. (a) the metric dimension of $(\mathbb{R}^n, \|\ \|)$ is equal to n (for any $n \in \mathbb{N}$);

(b) the metric dimension of (X, d^p) is equal to $\frac{1}{p}\mathrm{Dim}(X, d)$ (for any $p \in]0, 1[$);

(c) if f is a Lipschitz embedding of (X, d) into (Y, δ), we then have $\mathrm{Dim}(X, d) \leqslant \mathrm{Dim}(Y, \delta)$.

It follows from (a) and (c) that, if (X, d) admits a Lipschitz embedding into a space $(\mathbb{R}^n, \|\ \|)$, then the space (X, d^p) is of finite metric dimension for any $p \in]0, 1[$. We propose to establish a converse of sort to this observation. We will need two Lemmas.

2.3. Let (X, d) be a metric space, $b > 0$ a real number and $M \geqslant 2$ an integer; we define an (M, b)-*colouring of* (X, d) as any map $k : X \to \{1, ..., M\}$ such that $d(s, s') \leqslant b$ implies $k(s) \neq k(s')$, for all $s, s' \in X$ $(s \neq s')$.

As we know (Brooks [4]), every graph of degree $< M$ possesses an M-colouring. This can be reformulated as follows:

2.4. Lemma – *Let* (X, d) *be a metric space,* $b > 0$ *a real number and* $M \geqslant 2$ *an integer. Assume that we have* $|\{x \in X \mid d(x, s) \leqslant b\}| \leqslant M$, *for all* $s \in X$.

Then the space (X, d) *admits an* (M, b)-*colouring.*

Proof. – We equip X with a well-ordering by identifying it with the set of ordinals of cardinality $< \mathrm{Card}(X)$; for each $\alpha \in X$, we set $X_\alpha = \{\beta \in X \mid 0 \leqslant \beta < \alpha\}$.

(a) Let us fix $\alpha \in X$ and suppose that an (M, b)-colouring k_α of (X_α, d) is defined; let V be the set of values taken by k_α on $\{\beta \in X_\alpha \mid d(\beta, \alpha) \leqslant b\}$; we can choose an element m of $\{1, ..., M\}\backslash V$; we can then define an (M, b)-colouring $k_{\alpha+1}$ of $(X_{\alpha+1}, d)$ extending k_α by letting $k_{\alpha+1}(\alpha) = m$.

(b) This allows us to define inductively, for each $\alpha \in X$, an (M, b)-colouring k_α of (X_α, d) such that k_α extends k_β, for any $\alpha \in X$ and $\beta \in X_\alpha$.

The map $k : X \to \{1, ..., M\}$ which extends each of the k_α is then an (M, b)-colouring of (X, d). □

2.5 Lemma. – *Let (X, d) be a metric space, Y a 1-network of (X, d) (that is to say, a maximal 1-separated subset), $b > 8$ a real number and $M \geqslant 2$ an integer.*

We denote by $(e_1, ..., e_M)$ the canonical basis of $\mathbb{R}^M$. Let k be an (M, b)-colouring of (Y, d). Then the map $\varphi : s \to \sum_{y \in Y} (2 - d(s, y))^+ e_{k(y)}$ is a local (τ, A, B, c)-controlled embedding of (X, d) into $(\mathbb{R}^M, \| \ \|)$, for $c = b - 4$, $\tau = \frac{4}{c}$, $A = \sqrt{2}$ and $B = 4M$.

Proof. For each $y \in Y$, we denote by Δ_y the map $s \to (2 - d(s, y))^+$.

For each $s \in X$, we let $B_s = \{y \in Y \mid \Delta_y(s) \neq 0\}$.

(a) Let $s, t \in X$ with $4 < d(s, t) \leqslant b - 4$. Then the sets B_s and B_t are disjoint and the colouring k is injective on $B_s \cup B_t$. We then have:

$$\|\varphi(s) - \varphi(t)\|^2 \geqslant \mathrm{Sup}\{\Delta_y^2(s) + \Delta_z^2(t) \mid y \in B_s, z \in B_t\} \geqslant 2.$$

(b) Let $s, t \in X$. For each $y \in Y$, we have $|\Delta_y(s) - \Delta_y(t)| \leqslant 2(d(s, t) \wedge 1)$.

But we have $|B_s \cup B_t| \leqslant 2M$ (because k is injective on B_s and on B_t). So we have

$$\|\varphi(s) - \varphi(t)\| \leqslant \sum_{y \in B_s \cup B_t} |\Delta_y(s) - \Delta_y(t)| \leqslant 4M(d(s, t) \wedge 1).$$

□

We now arrive at the result that we had as our goal:

2.6. Proposition. – *Let (X, d) be a metric space of finite metric dimension and let $p \in]0, 1[$. Then there exists an integer $n \geqslant 0$ and a Lipschitz embedding of the space (X, d^p) into the Euclidean space $(\mathbb{R}^n, \| \ \|)$.*

Proof. – (a) We choose $C \in]0, +\infty[$ and $s \in]0, +\infty[$ such that the space (X, d) is (C, s)-homogeneous. Let τ be an element of $]0, 1[$; we let $c = \frac{4}{\tau}$, $b = c + 4$, $A = \sqrt{2}$, $M = C(2b)^s$ and $B = 4M$. Finally, we distinguish a point 0 in X.

(b) Let us fix $j \in \mathbb{Z}$. The space $(X, \tau^{-j} d)$ is also (C, s)-homogeneous. Let Y_j be a 1-network of $(X, \tau^{-j} d)$; we then have $|\{y \in Y_j \mid \tau^{-j} d(y, z) \leqslant b\}| \leqslant M$, for any $z \in Y_j$. The space $(Y_j, \tau^{-j} d)$ thus admits an (M, d)-colouring (see 2.4) and so there exists a local (τ, A, B, c)-controlled embedding φ_j of the space $(X, \tau^{-j} d)$ into the Euclidean space $(\mathbb{R}^M, \| \ \|)$ (see 2.5). By replacing φ_j by $\varphi_j - \varphi_j(0)$ if necessary, we can even assume that φ_j vanishes at 0.

(c) We have thus obtained, for each $j \in \mathbb{Z}$, a local (τ, A, B, c)-controlled embedding φ_j of $(X, \tau^{-j} d)$ into $(\mathbb{R}^M, \| \ \|)$ that vanishes at 0. Let $(v_j)_{j \in \mathbb{Z}}$ be a cyclic base of a Euclidean space $(F, \| \ \|)$. If the dimension of F is large enough, then the map $f : s \to \sum_{j \in \mathbb{Z}} \tau^{jp} \varphi_j(s) \otimes v_j$ is a Lipschitz embedding of the space (X, d^p) into the Euclidean space $(\mathbb{R}^M \otimes F, \| \ \|)$ (see 1.2). □

3 Generalised Koch curves

Our construction generalises, as we will see, that of the classical Koch curve [7]; it is what justifies our terminology.

3.1. Let $l \geqslant 2$ be an integer, η an element of $]0, 1[$, ψ an element of $[0, \pi[$, K a compact subset of $\mathbb{R}^n$ and a_0 and a_l two distinct points in K.

We define a *Koch chain with length* l*, scale* η*, flexibility* $\leqslant \psi$ *and mesh* (a_0, K, a_l) *in* $\mathbb{R}^n$ as any family $T = (T_0, ..., T_{l-1})$ of isometries of the Euclidean space satisfying the following properties:

(a) For each $r = 1, ..., l-1$, the set $\eta T_r(K)$ (denoted K_r) is contained in K;
(b) we have $\eta T_0(a_0) = a_0$ and $\eta T_{l-1}(a_l) = a_l$;
(c) for each $r = 1, ..., l-1$, the point $\eta T_r(a_0)$ (denoted a_r) is equal to $\eta T_{r-1}(a_l)$;
(d) for each $r = 1, ..., l-1$, we have

$$(x - a_r \mid y - a_r) + \|x - a_r\|\|y - a_r\| \cos\psi \leqslant 0,$$

for any $x \in K_{r-1}$ and $y \in K_r$;
(e) K_r and $K_{r'}$ are disjoint, for any $r, r' \in \{0, ..., l-1\}$ with $|r - r'| \geqslant 2$.

The sequence $(a_0, K_0, a_1, K_1, ..., a_{l-1}, K_{l-1}, a_l)$ is called the *support*, the points $a_0, ..., a_l$ the *vertices* and the sets $K_0, ..., K_{l-1}$ the *links* of the chain T.

Each Koch chain will allow the following construction:

3.2. Let $T = (T_0, ..., T_{l-1})$ be a Koch chain of length l, scale η, flexibility $\leqslant \psi$ and mesh (a_0, K, a_l) in $\mathbb{R}^n$.

(a) For each integer $j \geqslant 1$, we denote by D^l_j the set of real numbers t of the form $t = \sum_{i=1}^{j} r_i l^{-i}$ with $r_1, r_2, ..., r_j \in \{0, ..., l-1\}$. We denote by D^l the union of the sets D^l_j (for $j \geqslant 1$). For each $r = 0, ..., l-1$, we set $S_r = \eta T_r$.

(b) For each integer $j \geqslant 1$ and each element $t = \sum_{i=1}^{j} r_i l^{-i}$ of D^l_j (with $r_1, ..., r_j \in \{0, ..., l-1\}$), we set $f_j(t) = S_{r_1} S_{r_2} ... S_{r_j}(a_0)$.

(c) For each pair of integers j, k (with $j \leqslant k$), the map f_k obviously extends the map f_j (since we have $S_0(a_0) = a_0$, see 3.1b); we denote by f_T the map from D^l into $\mathbb{R}^n$ which is equal to f_j on D^l_j, for any integer $j \geqslant 1$; the set $\gamma_T = f_T(D^i)$ is then called *generalised Koch curve with respect to the chain* T.

There is an abuse of terminology in calling γ_T a curve; but we are going to show that the closure of γ_T is effectively a curve, and, more precisely, that the continuous extension of f_T to $[0, 1]$ is, for a certain $p \in]0, 1[$, a Lipschitz embedding of $([0, 1], \| \ \|^p)$ into $(\mathbb{R}^n, \| \ \|)$.

Before showing this, let us describe some examples of Koch chains:

3.3. Let us identify the Euclidean space $(\mathbb{R}^2, \| \ \|)$ with $\mathbb{C}$, and let us fix $\theta \in]0, \frac{\pi}{2}[$; we set $a_0 = 1 + e^{i\theta}$, $a_1 = 0$, $a_2 = -\bar{a}_0$ and we denote by K the triangle with vertices a_0, a_1 and a_2, by K_0 the triangle with vertices a_0, $a_0 - 1$ and a_1, and by K_1 the triangle with vertices a_1, $a_2 + 1$ and a_2. Clearly, there exists a Koch chain and a unique mesh (a_0, K, a_2) and having the sequence $(a_0, K_0, a_1, K_1, a_2)$ as its support; it is the chain T of length 2, scale $(2\cos\frac{\theta}{2})^{-1}$ and flexibility $\leqslant 2\theta$ in $\mathbb{R}^2$.

(a) In particular, for $\theta = \frac{\pi}{3}$, the curve γ_T is the set of points with dyadic parameter of the *classical Koch curve* (see [7]).

(b) For $\theta = \frac{\pi}{2}$ (which we had not allowed ourselves), the flexibility would no longer be bounded and γ_T would be the set of points with dyadic parameter of a Peano curve which fills K.

We are going to construct *quite a large class of Koch chains*:

3.4. Let $n \geqslant 1$ be an integer and $(e_1, ..., e_n)$ the canonical basis of $\mathbb{R}^n$.

(a) Let $\theta \in]0, \frac{\pi}{4}[$; we set $S(\theta) = \{x \in \mathbb{R}^n \mid \|x\| = tg\theta, (x \mid e_1) = 0\}$; and we denote by $D(\theta)$ the convex hull of $S(\theta) \cup \{e_1, -e_1\}$.

(b) We say that two points x and y in $\mathbb{Z}^n$ are adjacent if we have $\|x-y\| = 1$.

(c) Let X be a subset of $\mathbb{Z}^n$. A sequence $\gamma = (x_0, ..., x_l)$ of distinct points of X is called a *path of length l joining x_0 to x_l in X* if x_{r-1} and x_r are adjacent, for any $r = 1, ..., l$.

3.5 Lemma. – *Let $k_1, ..., k_n \geqslant 0$ integers. We set*

$$X = \prod_{i=1}^{n} \{0, ..., 2k_i\} \text{ and } a = 2\sum_{i=1}^{n} k_i e_i.$$

Then there exists a path γ of length $2l$ joining 0 to a in X, for any integer l satisfying $\sum_{i=1}^{n} k_i \leqslant l \leqslant \frac{1}{2}[\pi_{i=1}^{n}(2k_i+1)-1]$.

Proof. (Induction on $\sum_{i=1}^{n} k_i$). – Let $k_1, ..., k_n \geqslant 0$ be integers and let us suppose that the result is shown for every $k'_1, ..., k'_n$ with $\sum_{i=1}^{n} k'_i < \sum_{i=1}^{n} k_i$.

(a) We will assume $k_1 \geqslant 1$ (by permuting the coordinates if necessary) and set $m_2 = \sum_{i=2}^{n} k_i$, $M_2 = \frac{1}{2}[\prod_{i=2}^{n}(2k_i+1)-1]$, $m_1 = m_2 + k_1 - 1$, $M_1 = \frac{1}{2}[(2M_2+1)(2k_1-1)-1]$ and finally $m = m_1 + 1$, $M = M_1 + 2M_2 + 1$. We denote by X_1 (resp. X_2, resp. X_3) the set of points x of X such that $(x \mid e_1)$ is less than or equal to $2k_1 - 2$ (resp. is equal to $2k_1 - 1$, resp. is equal to $2k_1$).

(b) Let l be an integer satisfying $m \leqslant l \leqslant M$; so there exist integers l_1 and l_2 with $m_1 \leqslant l_1 \leqslant M_1$, $0 \leqslant l_2 \leqslant M_2$ and $l = l_1 + 2l_2 + 1$. So there exist (by the induction hypothesis) a path γ_1 of length $2l_1$ joining 0 and $a_1 = a - 2e_1$ in X_1, a point b_2 in X_2 and a path γ_2 of length $2l_2$ joining $a_2 = a - e_1$ and b_2 in X_2; so there also exists a path γ_3 of length $2l_2$ joining $b_3 = b_2 + e_1$ and a in X_3. Let γ be the path obtained by joining the paths γ_1, (a_1, a_2), γ_2, (b_2, b_3) and γ_3; it is the path of length $2l$ that we were looking for. □

3.6. Proposition. – *Let p be an element of $]0,1[$, n an integer strictly larger than $\frac{1}{p}$ and ψ an element of $]\frac{2\pi}{3}, \pi[$. Then there exists an integer $l \geqslant 2$ and a Koch chain of length l, scale l^{-p} and flexibility $\leqslant \psi$ in $\mathbb{R}^n$.*

Proof. (a) We take $\theta = \frac{\psi}{2} - \frac{\pi}{3}$. We choose a real number $\beta > 0$ such that the cube $[-2\beta, 2\beta]^n$ is contained in $\frac{1}{2}D(\theta)$. We choose an even integer $l \geqslant 2$ satisfying $2n(1+\beta)l^p \leqslant l \leqslant \beta^n l^{pn} - 1$ (this is possible because we have $p < 1 < pn$).
We set $\eta = l^{-p}$ and we denote by N the integer part of $\beta 1^p$.

(b) We set $b = N\sum_{i=1}^{n} e_i$, $K = \frac{1}{2\eta}D(\theta)$, $a_0 = \frac{1}{\eta}e_1$ and $X = \{-N, ..., N\}^n$. A sequence $\gamma = (y_0, ..., y_q)$ of distinct points of a subset Y of K will be called here a *trail of length q joining y_0 and y_q in Y* if $y_0, ..., y_q$ are consecutive vertices of a Koch chain of scale η, flexibility $\leqslant \psi$ and mesh $(a_0, K, -a_0)$.

(c) Let

$$Y^+ = \{x \in K \mid (x \mid e_1) \geqslant N + 1\}$$

and

$$Y^- = \{x \in K \mid (x \mid e_1) \leqslant -(N + 1)\}.$$

Clearly we can join a_0 and $b' = b + e_1$ by a trail γ^+ of length $r \leqslant \frac{n}{\eta} - 1$ in Y^+ (it is for this that we took $\psi > \frac{2\pi}{3}$). Likewise we can join $-b'$ and $-a_0$ by a trail γ^- of length r in Y^-. The integer $l - 2r - 2$ is thus even and satisfies

$$2nN \leqslant l - 2r - 2 \leqslant (2N+1)^n - 1.$$

So there exists (see Lemma 3.5) a path γ_0 of length $l-2r-2$ joining b and $-b$ in X. Let γ be the sequence obtained by joining the paths γ^+, (b',b), γ_0, $(-b,-b')$ and γ^-; it is a trail of length l joining a_0 and $-a_0$ in K, which proves the Proposition. □

Having obtained some examples of Koch chains, we are now going to show that each chain of length l and scale l^{-p} in $\mathbb{R}^n$ defines a Lipschitz embedding of $([0,1],\| \ \|^p)$ into $(\mathbb{R}^n,\| \ \|)$.

Here are some preliminary observations:

3.7. (*a*) Let T be a Koch chain of length l, scale η and mesh (a_0,K,a_l) in $\mathbb{R}^n$. Let f_T be the map from D^l into $\mathbb{R}^n$ which it defines (see 3.2). Then the points $f_T(0)$, $f_T(\frac{1}{l})$, ..., $f_T(\frac{l-1}{l})$, a_l are the vertices of the chain T.

(*b*) Let $k \geqslant 2$ be an integer. Then the points

$$f_T(0), f_T(l^{-k}), f_T(2l^{-k}), ..., f_T((l^k-1)l^{-k}),$$

a_l are the vertices of a chain $T(k)$ of length l^k, scale η^k, mesh (a_0,K,a_l) and which satisfies $f_{T(k)} = f_T$ (observe that we have $D^{l^k} = D^l$).

(*c*) Let $x,y \in D^l$ with $|x-y| > \frac{1}{2l}$; then $f_T(x)$ and $f_T(y)$ belong to non-consecutive links of the chain $T(3)$ (because we have $\frac{1}{2l} \geqslant 2l^{-3}$).

(*d*) The quantity $A_T = \mathrm{Inf}\{\|f_T(x)-f_T(y)\| \mid |x-y| > \frac{1}{2l}\}$ is thus zero. Furthermore, we denote by B_t the diameter of K.

(*e*) Let $s,t \in D^l$ (with $s<t$); let $j \in \mathbb{N}$ such that $]s,t[$ do not contain any element of D^l_j. We set $\bar{s}^j = l^j(s-z)$ and $\bar{t}^j = l^j(t-z)$, where z is the largest element of D^l_j smaller than or equal to s. Then we have

$$\|f_T(s)-f_T(t)\| = l^{-jp}\|f_T(\bar{s}^j)-f_T(\bar{t}^j)\| \leqslant B_T l^{-jp}.$$

If, moreover, we have $|s-t| > \frac{1}{2}l^{-(j+1)}$, then we also have $\|f_T(s)-f_T(t)\| \geqslant A_t l^{-jp}$.

3.8. Proposition. – *Let p be an element of $]0,1[$ and let $T=(T_0,...,T_{l-1})$ be a Koch chain of length l, scale l^{-p}, flexibility $\leqslant \psi$ and mesh (a_0,K,a_l) in $\mathbb{R}^n$. Let f_T be the map from D^l into $\mathbb{R}^n$ which it defines (see 3.2).*

Then f_T is a Lipschitz embedding of $(D^l,\| \ \|^p)$ into $(\mathbb{R}^n,\| \ \|)$. More precisely, let A_T and B_T be the quantities defined in 3.6d: we then have

$$A_T|x-y|^p \sin\left(\psi \vee \frac{\pi}{2}\right) \leqslant \|f_T(x)-f_T(y)\| \leqslant 2B_T l^p|x-y|^p,$$

for any $x,y \in D^l$.

Proof. – Let $x,y \in D^l$ (with $x<y$) and let $j \geqslant 0$ be the integer satisfying $l^{-(j+1)} < |x-y| \leqslant l^{-j}$. We distinguish two cases:

(*a*) If $]x,y[$ does not contain any element of D^l_j, we have

$$A_T l^{-jp} \leqslant \|f_T(x)-f_T(y)\| \leqslant B_T l^{-jp}$$

(see 3.6e) and so

$$A_T|x-y|^p \leqslant \|f_T(x)-f_T(y)\| \leqslant B_T l^p|x-y|^p.$$

(b) Otherwise, $]x, y[$ contains a unique element a of D_j^l; by replacing T with the reverse chain if necessary, we can assume that we have $\frac{1}{2}l^{-(j+1)} < |x - a| \leqslant l^{-j}$ and $|a - y| \leqslant l^{-j}$; so we have

$$\|f_T(x) - f_T(a)\| \sin\left(\psi \vee \frac{\pi}{2}\right) \leqslant \|f_T(x) - f_T(y)\|$$
$$\leqslant \|f_T(x) - f_T(a)\| + \|f_T(a) - f_T(y)\|$$

(the first inequality comes from the fact that the chain T is of flexibility $\leqslant \psi$; see 3.1d); so we have $A_T l^{-jp} \sin\left(\psi \vee \frac{\pi}{2}\right) \leqslant \|f_T(x) - f_T(y)\| \leqslant 2B_T l^{-jp}$ (see 3.6e), which implies that

$$A_T|x - y|^p \sin\left(\psi \vee \frac{\pi}{2}\right) \leqslant \|f_T(x) - f_T(y)\| \leqslant 2B_T l^p |x - y|^p.$$

□

4 Rank of the space $([0,1]^k, \| \ \|^p)$

4.1 (a) Let (X, d) be a metric space. We define the *rank of* (X, d) (denoted by $\mathrm{rg}(X, d)$) as the smallest integer $n \geqslant 0$ such that there exists a Lipschitz embedding of (X, d) into the Euclidean space $(\mathbb{R}^n, \| \ \|)$.

(b) Let f be a Lipschitz embedding of (X, d) into the metric space (Y, δ); we define the *distortion of* f (denoted by $\Delta(f)$) as the lower bound of the real numbers λ such that there exists $A \in]0, +\infty[$ for which we have $Ad(x, y) \leqslant \delta(f(x), f(y)) \leqslant Ae^\lambda d(x, y)$, for any $x, y \in X$.

(c) If n is an integer $\geqslant \mathrm{rg}(X, d)$, we define the n*-distortion of* (X, d) (denoted by $\Delta_n(X, d)$) as the lower bound of $\Delta(f)$ over every Lipschitz embedding f of (X, d) into the Euclidean space $(\mathbb{R}^n, \| \ \|)$.

We will not try to evaluate the n-distortion of $([0, 1]^k, \| \ \|^p)$ here; let us however describe some questions that arise.

4.2 If (X, d) embeds isometrically, i.e. with distortion 0, into an infinite-dimensional Hilbert space (which is the case for the space $([0, 1]^k, \| \ \|^p)$), then we can expect that the n-distortion of (X, d) tends to 0 when n tends to $+\infty$. Thus Kahane [6] showed, in response to a question by the author, that the n-distortion of $([0, 1], \sqrt{\| \ \|})$ is smaller than or equal to $0\left(\frac{1}{n}\right)$.

On the other hand, we are going to evaluate quite precisely the rank of $([0, 1]^k, \| \ \|^p)$. Here are first some obvious preliminary remarks:

4.3 (a) Let d and δ be two metrics on a set X. We assume that the identity is a Lipschitz embedding of (X, d) into (X, δ) (we say in this case that d and δ are *Lipschitz equivalent*). Then (X, d) and (X, δ) have the same rank.

(b) Let (X, d) and (Y, δ) be two metric spaces. We equip $X \times Y$ with the *direct sum metric* $d \oplus \delta : (x, y), (x', y') \to d(x, x') + \delta(y, y')$. The rank of $(X \times Y, d \oplus \delta)$ is then smaller than or equal to $\mathrm{rg}(X, d) + \mathrm{rg}(Y, \delta)$.

(c) Let (X, d) be a metric space. We then have $\mathrm{Dim}(X, d) \leqslant \mathrm{rg}(X, d)$.

Let us now summarise the results obtained in Section 3:

4.4 Proposition. – *Let* $k \geqslant 1$ *be an integer and* p *an element of* $]0, 1[$. *Let* n *be the smallest integer* $> \frac{1}{p}$. *Then the space* $([0, 1]^k, \| \ \|^p)$ *is of rank* $\leqslant kn$.

Proof. – (*a*) We have shown, in 3.6, the existence of an integer $l \geqslant 2$ and a Koch chain of length l, scale l^{-p} and flexibility $\leqslant \frac{3\pi}{4}$ in $\mathbb{R}^n$. As a result, there exists (see 3.8) a Lipschitz embedding of $(D^l, \| \ \|^p)$ into $(\mathbb{R}^n, \| \ \|)$. Continuously extending this embedding, we obtain a Lipschitz embedding of $([0,1], \| \ \|^p)$ into $(\mathbb{R}^n, \| \ \|)$. So the space $([0,1], \| \ \|^p)$ is of rank $\leqslant n$.

(*b*) Let us denote by d (resp. δ) the metric $x, y \to \|x-y\|^p$ *on* $[0,1]^k$ (resp. on $[0,1]$). The metric d is Lipschitz equivalent to $\delta \oplus ... \oplus \delta$ (k times). The remark 4.3*b* thus shows that $([0,1]^k, \| \ \|^p)$ is of rank $\leqslant nk$. □

4.5 As a result of 2.2, the metric dimension of $([0,1]^k, \| \ \|^p)$ is equal to $\frac{k}{p}$. Taking into account 4.3c, we see that $([0,1]^k, \| \ \|^p)$ is of rank $\geqslant \frac{k}{p}$.

The rank of $([0,1], \| \ \|^p)$ (for $p \in]0,1[$) is thus the smallest integer $> \frac{1}{p}$, if $\frac{1}{p}$ is not an integer. We propose to prove that this result remains true even if $\frac{1}{p}$ is an integer (different from 1). More generally, we are going to establish that $([0,1]^k, \| \ \|^p)$ is of rank $> \frac{k}{p}$ for any integer $k \geqslant 1$ and $p \in]0,1[$.

This will result from three lemmas.

4.6. (*a*) A symmetric kernel $d : X^2 \to]0, +\infty[$ which is zero on and only on the diagonal is called a *pseudometric on* X if there exists a number $a \in [1, +\infty[$ such that we have $d(x,y) \leqslant a(d(x,z) + d(z,y))$ for any $x, y, z \in X$ (when we want to be more precise, we say that d is an *a-pseudometric on* X). We then equip the space (X, d), which we call a *pseudometric space*, with the topology generated by the "open" balls with respect to the pseudometric d.

(*b*) Let (X, d) be a pseudometric space; for each open subset U of (X, d), we denote by $\tau(U)$ the *diameter of* (U, d), i.e. the quantity $\mathrm{Sup}\{d(u,v) \mid u, v \in U\}$. We then define an outer measure μ_d on X in the following way: for each subset A of X and each $\epsilon > 0$, we set

$$\mu_{d,\epsilon}(A) = \mathrm{Inf}\{\sum_{i \in \mathbb{N}} \tau(U_i) \mid (U_i)_{i \in \mathbb{N}} \text{ covering of } A \text{ by open sets of diameter } \leqslant \epsilon\}$$

and

$$\mu_d(A) = \mathrm{Sup}_{\epsilon > 0} \mu_{d,\epsilon}(A)$$

(so the measure μ_d is the outer measure on X obtained by Method II of Rogers [8] p.27 from the pre-measure τ).

We will say that μ_d is the *Hausdorff measure on* (X, d).

4.7 Lemma – *Let* (X, d) *and* (Y, δ) *be two pseudometric spaces; let* $A, B \in]0, +\infty[$. *Let* f *be a map from* X *into* Y *satisfying* $Ad(x,x') \leqslant \delta(f(x), f(x')) \leqslant Bd(x,x')$ *for any* $x, x' \in X$. *We then have* $A\mu_d(X) \leqslant \mu_\delta(f(X)) \leqslant B\mu_d(X)$.

Proof. When d and δ are metrics, it is a particular case of Theorem 29 in [8]. Moreover, the proof of this theorem clearly remains valid (see [8] p.54) even if d and δ are not metrics. □

4.8 Every power of a metric is a pseudometric; in addition, we can show ([1] Lemma 1.14) that every pseudometric is Lipschitz equivalent to a power of a metric.

4.9 Lemma. – *Let* (X, d) *and* (Y, δ) *be two metric spaces,* x_0 *a point in* X *and* y_0 *a point in* Y*; let* $A, B \in]0, +\infty[$. *We assume that the closed balls of* (Y, δ) *are compact and that, for each finite subset* F *of* X *containing* x_0*, there exists* *an (A,B)-Lipschitz embedding g_F of (F,d) into (Y,δ) satisfying $g_F(x_0)=y_0$. Then there exists an (A,B)-Lipschitz embedding of (X,d) into (Y,δ).*

Proof. For each finite subset F of X containing x_0, we define a map f_F from X into Y by letting $f_F(x)=g_F(x)$ if x belongs to F, and $f_F(x)=y_0$ otherwise. We then let $f(x)=\lim_{F,\mathscr{U}} f_F(x)$ (for each $x\in X$), where $\mathscr{U}$ is an ultrafilter finer than the filter of inclusion on the set of every finite subsets of X containing x_0 (we observe that $f_F(x)$ is, for any F, an element of the closed ball centred at y_0 with radius $Bd(x_0,x)$; it is the compactness of this ball which ensures the existence of $f(x)$).

The map f is the embedding that we were looking for. □

4.10 Let $\varepsilon\in]0,+\infty[$. A subset T of a metric space (X,d) is said to be ε-*dense in* (X,d) if, for each $x\in X$, there exists $t\in T$ with $d(x,t)<\varepsilon$.

4.11. Lemma. – *Let F be an α-separated subset and G an ε-dense subset of a metric space (X,d). Then there exists a map $h:F\to G$ satisfying $\left(1-\frac{2\varepsilon}{\alpha}\right)d(x,y)\leqslant\delta(h(x),h(y))\leqslant\left(1+\frac{2\varepsilon}{\alpha}\right)d(x,y)$ for any $x,y\in F$.*

Proof. As G is ε-dense, we can choose, for each $x\in F$, a point $h(x)$ in G with $d(x,h(x))<\varepsilon$. Let us fix $x,y\in F$ (with $x\neq y$); we then have:

$$\begin{aligned} d(h(x),h(y)) &\leqslant d(x,y)+d(x,h(x))+d(y,h(y)) \\ &\leqslant d(x,y)+2\varepsilon\leqslant\left(1+\frac{2\varepsilon}{\alpha}\right)d(x,y) \end{aligned}$$

and

$$\begin{aligned} d(x,y) &\leqslant d(h(x),h(y))+d(x,h(x))+d(y,h(y)) \\ &\leqslant d(h(x),h(y))+\frac{2\varepsilon}{\alpha}d(x,y). \end{aligned}$$

Hence the map h is the map that we were looking for. □

Now, here is the result that we had in sight:

4.12. Proposition. – *Let $k\geqslant 1$ be an integer and p an element of $]0,1[$. Denote by $r(k,p)$ the rank of the metric space $([0,1]^k,\|\ \|^p)$. We then have $m\leqslant r(k,p)\leqslant kn$, where m is the smallest integer $>\frac{k}{p}$, and n the smallest integer $>\frac{1}{p}$.*

In particular, the rank of $([0,1],\|\ \|^p)$ is the smallest integer $>\frac{1}{p}$.

Proof. – (a) We have already shown the inequalities $r(k,p)\leqslant kn$ (see 4.4) and $r(k,p)\geqslant\frac{k}{p}$ (see 4.5). Hence it remains to establish the inequality $r(k,p)>\frac{k}{p}$, in the case where $\frac{k}{p}$ is an integer.

(b) For that, we are going to assume that there exists an (A,B)-Lipschitz embedding f of $([0,2]^k,\|\ \|^p)$ into $(\mathbb{R}^q,\|\ \|)$ (with $\frac{k}{p}=q\in\mathbb{N}$) and show that this leads to a contradiction; this will establish the Proposition.

(c) Let f be the embedding whose existence we assumed in (b). So we have $A^q\|x-y\|^k\leqslant\|f(x)-f(y)\|^q\leqslant B^q\|x-y\|^k$, for any $x,y\in[0,2]^k$. Lemma 4.7 (applied to the pseudometrics $d:x,y\to\|x-y\|^k$ on $[0,2]^k$ and $\delta:s,t\to\|s-t\|^q$ on $\mathbb{R}^q$) implies that the set $f\left(\left[\frac{1}{2},\frac{3}{2}\right]^k\right)$ has non-zero Lebesgue measure in $\mathbb{R}^q$ and so possesses (by the Lebesgue differentiation theorem) a density point $t_0 = f(x_0)$.

(d) For each $\beta \in]0, \frac{1}{2}[$ and each $x \in [-1,1]^k$, we set

$$f_\beta(x) = \beta^{-p}(f(x_0 + \beta x) - f(x_0)).$$

For each $\beta \in]0, \frac{1}{2}[$ the map f_β thus defined is an (A, B)-Lipschitz embedding of $([-1,1]^k, \| \ \|^p)$ into $(\mathbb{R}^q, \| \ \|)$ and we have:

$$\beta^k \lambda(f_\beta([-1,1]^k) \cap S(0,A)) \\ = \lambda(f(x_0 + \beta[-1,1]^k) \cap S(t_0, A\beta^p)) \geqslant \lambda(f([0,2]^k) \cap S(t_0, A\beta^p))$$

(where we denoted by λ the Lebesgue measure on $\mathbb{R}^q$ and by $S(z,r)$ the closed ball centred at z and with radius r in the space $(\mathbb{R}^q, \| \ \|)$).

(e) As t_0 is a density point in $f\left(\left[\frac{1}{2}, \frac{3}{2}\right]^k\right)$, the inequality that we established in (d) shows that we have

$$\lim_{\beta \to 0} \lambda(f_\beta([-1,1]^k) \cap S(0,A)) = \lambda(S(0,A)).$$

Hence, for each $\varepsilon > 0$, we can choose a number $\beta(\varepsilon) \in]0, \frac{1}{2}[$ such that $G_\varepsilon = f_{\beta(\varepsilon)}([-1,1]^k) \cap S(0,A)$ is ε-dense in the space $(S(0,A), \| \ \|)$.

Moreover, $g_\varepsilon = f^{-1}_{\beta(\varepsilon)}$ is a $\left(\frac{1}{B}, \frac{1}{A}\right)$-Lipschitz embedding of the space $(G_\varepsilon, \| \ \|)$ into the space $([-1,1]^k, \| \ \|^p)$.

(f) Let F be a finite subset of $S(0,A)$ containing 0; so there exists $\alpha > 0$ such that F is α-separated in $(S(0,A), \| \ \|)$. We fix $\varepsilon = \frac{\alpha}{4}$. So there exists (by Lemma 4.11) a $\left(\frac{1}{2}, \frac{3}{2}\right)$-Lipschitz embedding h_F of $(F, \| \ \|)$ into $(G_\varepsilon, \| \ \|)$, and we can assume $h_F(0) = 0$ (because 0 belongs to G_ε). The map $g_F = g_\varepsilon \circ h_F$ is thus a $\left(\frac{1}{2B}, \frac{3}{2A}\right)$-Lipschitz embedding of the space $(F, \| \ \|)$ into the space $([-1,1]^k, \| \ \|^p)$, and it satisfies $g_F(0) = 0$.

(g) Hence there exists (by Lemma 4.9) a $\left(\frac{1}{2B}, \frac{3}{2A}\right)$-Lipschitz embedding of the space $(S(0,A), \| \ \|)$ into the space $([-1,1]^k, \| \ \|^p)$, in contradiction with the fact that the topological dimension of $S(0,A)$ is equal to $q > k$. This is the contradiction that we were looking for. □

The inequality $r\left(1, \frac{1}{2}\right) > 2$ could have been proved by using the following result of Besicovitch and Schoenberg:

4.13. [2] Let f be a continuous and injective map from $[0,1]$ into $\mathbb{R}^2$. We then have $\mathrm{Inf}\{\sum_{i=1}^{j} \|f(x_i) - f(x_{i-1})\|^2\} = 0$. where the infimum is taken on all the partitions $0 = x_0 < x_1 < x_2 < ... < x_{j-1} < x_j = 1$ of the segment $[0,1]$.

Likewise the inequality $r\left(1, \frac{1}{q}\right) > q$ could have been established by using an extension (due to Y. Katznelson, not published) of 4.13 to Jordan curves $f : [0,1] \to \mathbb{R}^q$, the q-variation then replacing the 2-variation.

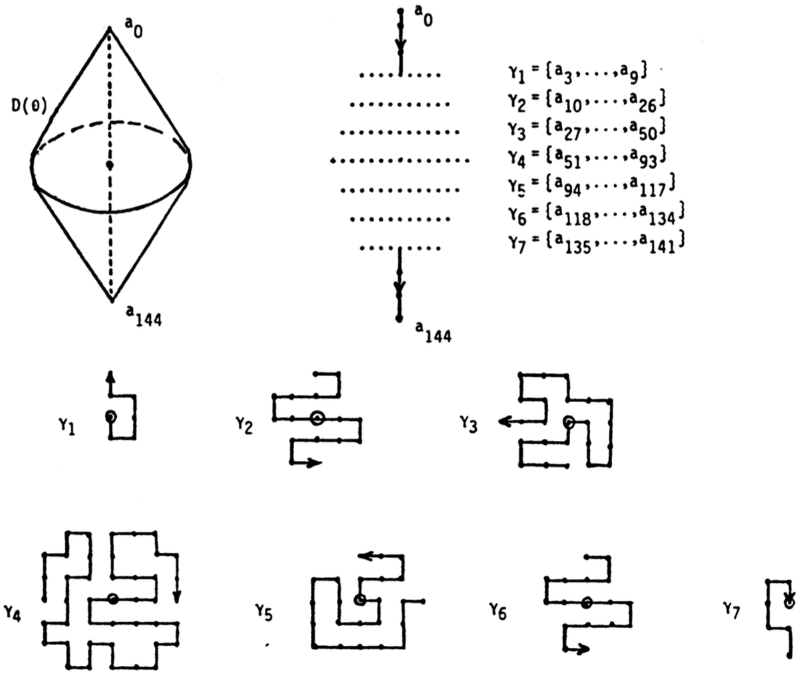
BIBLIOGRAPHY

[1] Assouad (P.). – Espaces métriques, plongements, facteurs, Thèse (Orsay, 1977)
[2] Besicovitch (A.S.), Schoenberg (I.J). – On Jordan arcs and Lipschitz classes of functions defined on them. Acta Math. 106 (1961), 113-136.
[3] Bouligand (G.). – Ensembles impropres et ordre dimensionnel, Bull. Sci. Math.

52 (1928), 320-344 and 361-376.
[4] Brooks (R.L.). – On colouring the nodes of a network, Proc. Cambridge Phil. Soc. 37 (1941), 194-197.
[5] Glaeser (G.). – Étude de quelques algèbres tayloriennes, J. Analyse Math. 6(1958), 1-124.
[6] Kahane (J.P.). – Hélices et quasi-hélices, Math. Analysis and Appl. (L. Nachbin ed.), 417-433, Academic Press, 1981, New York.
[7] Von Koch (H.). – Une méthode géométrique élémentaire pour l'étude de certaines questions de la théorie des courbes planes, Acta Math. 30 (1906), 145-174.
[8] Rogers (C.A.). – Hausdorff measures, Cambridge University Press (1970).

APPENDIX. A curve leading to an embedding of $([0,1], \sqrt{\|\ \|})$ *into* $\mathbb{R}^3$*:*

Let $\theta = \mathrm{Arctg}\frac{2}{3}$; we consider a Koch chain T of length 144, scale $\frac{1}{12}$, flexibility $\leqslant \frac{\pi}{2}+2\theta$ and mesh $(a_0, D(\theta), a_{144})$ in $\mathbb{R}^3$, whose support $(a_0, K_0, a_1, ..., a_{143}, K_{143}, a_{144})$ is given by the following diagram:



A crude calculation (by hand) shows that there exists a real number $A > 0$ such that we have $A\sqrt{|s-t|} \leqslant \|f_T(s) - f_T(t)\| \leqslant 2184A\sqrt{|s-t|}$, for any $s, t \in [0,1]$.