Hilbert Subspaces of Topological Vector Spaces and Associated Kernels (Reproducing Kernels)

by

LAURENT SCHWARTZ,

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Translated by Junhyung Park[1]

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[1]This is not authoritative translation, and I do not claim any credit for the mathematical content of this document. Please send any corrections to `junhyung.park@tuebingen.mpg.de`.

Introduction.

The kernel associated to a Hilbert space was introduced by S. Bergman, for the space of holomorphic functions on an open subspace of $\mathbb{C}^n$; by himself and with other authors, he systematically applied it to the study of certain problems related to holomorphic functions, while he introduced other kernels relative to other spaces and used them in problems of partial derivatives (see Bergman [1], Bergman and Schiffer [1]). It was Aronszajn who showed that this was a particular case of a general situation: one can endow a "reproducing" kernel on any Hilbert subspace of a space of functions on a set X; he studied the relationship between the kernel and the space, introducing the operations of scalar multiplication and addition, and the order structure. Furthermore, he also showed close links between the kernels and the solution of certain problems in the limit of partial derivatives (see Aronszajn). Our goal is to extend the formalism further. If E is any locally convex, quasi-complete Hausdorff topological vector space, one can define a set $\mathrm{Hilb}(E)$ of Hilbert subspaces of E, endowed with a structure of a "salient convex cone", defined by the operations indicated above; one can moreover introduce the space $\mathscr{L}^+(E)$ of non-negative kernels relative to E, a non-negative kernel being a linear, weakly continuous, non-negative map from $\bar{E}'$ into E; it is also endowed with the structure of a salient convex cone. And there exists a canonical isomorphism between these two cones.

§0 introduces general background on spaces and conjugate spaces, which could or should appear in elementary literature on vector spaces, as well as the notion of the conjugate and the adjoint of a continuous linear map.

§1 introduces the pre-Hilbert and Hilbert subspaces of a topological vector space, and the important notion of the completion of a pre-Hilbert subspace; the notion of Q-completion is less necessary to the understanding of the subsequent material.

§2 introduces the cone $\mathrm{Hilb}(E)$ of Hilbert subspaces of E, and formally defines the three fundamental structures: multiplication by a non-negative scalar, addition and order relation.

§3 introduces kernels relative to E, and some topological vectorial considerations.

§4 defines the non-negative kernel associated to a Hilbert subspace of E, or the reproducing kernel, or the canonical map from $\mathrm{Hilb}(E)$ into $\mathscr{L}^+(E)$. Proposition 6 can serve as the definition. Proposition 8 will play a fundamental role in the subsequent development: it recovers the Hilbert space from its kernel, and thereby shows that the canonical map $\mathrm{Hilb}(E) \to \mathscr{L}^+(E)$ is injective.

§5 establishes the bijectivity of this map: any non-negative kernel is the kernel associated to a unique Hilbert subspace (Proposition 10).

§6 establishes the isomorphism between the two cone structures of $\mathrm{Hilb}(E)$ and $\mathscr{L}^+(E)$: the previously defined bijection preserves the three fundamental structures.

§7 gives some consequences of this isomorphism. In particular, Proposition 14 establishes the existence and uniqueness of the difference of two Hilbert subspaces, $\mathscr{H}_1 - \mathscr{H}_2$, if $\mathscr{H}_1 \geqq \mathscr{H}_2$. Propositions 18 and 19 treat infinite sums of Hilbert subspaces and of kernels, and hence the Hilbert bases of Hilbert subspaces (Corollary 5 of Proposition 19), and Proposition 20 is concerned with integrals (measurable sums) of Hilbert subspaces and of kernels.

§8 studies the effect of a continuous linear map. If E and F are two topological vector spaces, and u a continuous linear map from E into F, then u defines two equivalent maps, one from $\mathrm{Hilb}(E)$ to $\mathrm{Hilb}(F)$ and the other from $\mathscr{L}^+(E)$ to $\mathscr{L}^+(F)$ (Proposition 21). From this, one can deduce the functorial character of the maps Hilb and $\mathscr{L}^+$ (page 50).

§9 considers the particular case studied by Bergman and Aronszajn, where E is a space of complex functions on a set X, equipped with the topology of pointwise convergence. The propositions shown here often simply reproduce those of Aronszajn. To see the connections between this particular case and the general case, one can read page 52. Proposition 27 reconstructs Bergman's kernel on a complex analytical manifold.

§10 studies a completely novel situation, drawn from the theory of distributions. The space $\mathscr{D}$, whose dual $\mathscr{D}'$ is the space of distributions, is at the same time a subspace of $\mathscr{D}'$. Thus we study the situation where $\bar{E}'$ (that is to say, $\mathscr{D}$) is a subspace of E (that is to say, $\mathscr{D}'$). In general, we define the notion of normal subspaces of E, which corresponds to that of spaces of normal distributions. Thus, if $\mathscr{H}$ is a normal Hilbert subspace of E, its conjugate dual $\bar{\mathscr{H}}'$ is again a subspace of E; their kernels H and H' are related in a very interesting way: they are inverses of each other in some sense, and each of the two subspaces can be reconstructed with the kernel of the other (Propositions 28 to 31). We deduce some applications of this on limit problems of Dirichlet and von Neumann type (page 73).

§11 studies in detail some applications in classical potential theory: charges and finite-energy potentials, balayage, restriction to an open subspace and Green's operator. The language of Hilbert subspaces and associated kernels are very fertile here.

§12 attempts a generalisation to Hermitian spaces (with a non-positive metric) and associated Hermitian kernels. Here, we encounter severe difficulties. It seems that, no matter what method we employ, a Hermitian kernel is no longer associated to a single Hermitian subspace, but a class of Hermitian subspaces; the canonical map $\mathrm{Herm}(E) \to \mathscr{L}^h(E)$, which extends the map $\mathrm{Hilb}(E) \to \mathscr{L}^+(E)$, is surjective, but is no longer injective. Nevertheless, it could be that this is not a monstrosity, but an interesting novelty.

§13 gives sufficient conditions, and in some cases necessary and sufficient conditions (Corollary of Proposition 41), for a Hermitian kernel to come from a single Hermitian subspace.

The present work was preceded by other publications. It may be of advantage to read those first*.

Furthermore, in the future, we will publish works on applications in differential operators (see Proposition 34, for example) and unitary representations of Lie groups and distributions of positive-type on these groups.

*See Schwartz [4] and [5].

§0. Spaces and conjugate spaces

Conjugate spaces. Let E be a locally convex Hausdorff topological vector space[(1)] over $\mathbb{C}$. The conjugate space of E consists of a locally convex Hausdorff topological vector space $\bar{E}$ and an anti-isomorphism from E onto $\bar{E}$, which we generally call "conjugation" and denote by $e \mapsto \bar{e}$. Naturally, E is the conjugate space of $\bar{E}$, corresponding to the reciprocal anti-isomorphism $\bar{e} \mapsto e$. There exists only one conjugate space, up to isomorphism. In other words, if $\bar{E}_1$ and $\bar{E}_2$ are two conjugate spaces of E, corresponding to anti-isomorphisms $e \mapsto \bar{e}_1$ and $e \mapsto \bar{e}_2$, there exists a unique isomorphism i from $\bar{E}_1$ onto $\bar{E}_2$, satisfying $i\bar{e}_1 = \bar{e}_2$ for all e in E. In practice, we encounter many models of conjugate spaces:

Examples.

1°) Let us place on E the original structure of an additive group, and a "hacked" multiplication law by complex scalars, denoted $e \mapsto \lambda \times e$, such that

$$\lambda \times e = \bar{\lambda} e. \tag{0.1}$$

Equipped with these laws, and with the original topology, E becomes a new topological vector space over $\mathbb{C}$, denoted $\bar{E}$. Here, $\bar{E}$ is a conjugate space of E, if we take the identity $e \mapsto e$ for the conjugation. Every space E possesses a conjugate space, and only one up to isomorphism.

2°) Often, a topological vector space E is equipped with an anti-automorphism $e \mapsto \bar{e}$. We can then take $\bar{E} = E$, the conjugation being the given anti-automorphism. For the conjugate space of $\bar{E}$, we can take either E with conjugation $\bar{e} \mapsto e$ or E with conjugation $\bar{e} \mapsto \bar{\bar{e}}$: the canonical isomorphism i between these two conjugate spaces, defined at the beginning, is thus $e \mapsto \bar{\bar{e}}$. Usually, the given anti-automorphism will be an anti-involution, i.e. we have $\bar{\bar{e}} = e$, so that this technicality is not an issue.

For example, if f is a complex function on a set X, we define $\bar{f}$ by $\bar{f}(x) = \overline{f(x)}$, and many function spaces in analysis admit the anti-involution $f \mapsto \bar{f}$, and are thus conjugate spaces of themselves. The same holds true for current spaces and distribution spaces[(2)].

3°) Let $\bar{E}$ be a conjugate space of E. Let E' be its dual (space of continuous linear forms on E); we will denote by $\langle e, e' \rangle$ the scalar product of $e \in E$ and $e' \in E'$. For $e' \in E'$, $\bar{e} \mapsto \overline{\langle e', e \rangle}$ is a continuous linear form on $\bar{E}$, thus defining an element $\overline{e'}$ in $\bar{E}'$. So $e' \mapsto \overline{e'}$ is an antilinear bijection from E' onto $\bar{E}'$; it is a homeomorphism, so an anti-isomorphism, if we equip E' and $\bar{E}'$ with corresponding topologies (weak, strong, etc... If $\mathfrak{S}$ is a family of subsets of E and $\bar{\mathfrak{S}}$ the corresponding family of conjugated subsets in $\bar{E}$, we could take the $\mathfrak{S}$-convergence and $\bar{\mathfrak{S}}$-convergence on E' and $\bar{E}'$ respectively[(3)]). We can thus take $\bar{E}'$ as the conjugate space $\overline{E'}$ of

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[(1)] For most of the definitions concerning topological vector spaces, we will refer to Bourbaki [1]. A topological vector space is locally convex if the origin has a neighbourhood basis consisting of convex sets. A map u is antilinear if $u(x+y) = u(x) + u(y)$ and if, for all scalars λ, we have $u(\lambda x) = \bar{\lambda} u(x)$. An anti-isomorphism is an antilinear homeomorphism.

[(2)] For distributions, we refer to Schwartz [1], and for currents, to de Rham [1].

[(3)] See Bourbaki [1], Chapter III, §3, n°1.

E' with the conjugation $e' \mapsto \overline{e'}$; so we have the rule $\bar{E}' = \overline{E'}$ and

$$\langle \overline{e'}, \bar{e} \rangle = \overline{\langle e', e \rangle}. \tag{0.2}$$

Taking into account the definitions of the transpose[(4)] and the contragredient of an antilinear map, $e' \mapsto \overline{e'}$ is the contragredient of $e \mapsto \bar{e}$ according to (0.2).

For example, in 1°), $\bar{E}'$ is nothing else but the anti-dual of E (the space of continuous antilinear forms on E). In 2°), $\bar{E}'$ is nothing else but E', and $e' \mapsto \bar{e}'$ is an anti-involution on E', contragredient of the anti-involution of E. Hence, from the anti-involution $\phi \mapsto \bar{\phi}$ on the space $\mathscr{D}(X)$ of compactly supported C^∞ functions on an open subset X of $\mathbb{R}^n$ (or compactly supported C^∞ differential forms on a manifold X of C^∞-class), we deduce the anti-involution $T \mapsto \bar{T}$ on the space $\mathscr{D}'(X)$ of distributions (or currents) on X, by

$$\langle \bar{T}, \bar{\phi} \rangle = \overline{\langle T, \phi \rangle}; \tag{0.3}$$

this involution makes $\mathscr{D}'(X)$ the conjugate space of itself.

It would sometimes be advantageous to introduce sesquilinear scalar products[(5)] on $E \times \bar{E}'$, $E' \times \bar{E}$, $\bar{E} \times E'$ and $\bar{E}' \times E$:

$$\begin{cases} (e \mid \bar{e}') = (e' \mid \bar{e}) = \langle e, e' \rangle (= \langle e', e \rangle) \\ (\bar{e} \mid e') = (\bar{e}' \mid e) = \overline{\langle e, e' \rangle}. \end{cases} \tag{0.4}$$

4°) Let $\mathscr{H}$ be a Hilbert space; we will denote by $(e \mid f)_{\mathscr{H}}$ its sesquilinear inner product. For its conjugate $\bar{\mathscr{H}}$, we can take its dual $\mathscr{H}'$, the conjugation being the canonical anti-isomorphism from $\mathscr{H}$ onto $\mathscr{H}'$; if $h \in \mathscr{H}$, $\bar{h}$ will then be the element of $\mathscr{H}' = \bar{\mathscr{H}}$ defined by

$$\langle k, \bar{h} \rangle = (k \mid h)_{\mathscr{H}}, \qquad \forall k \in \mathscr{H}. \tag{0.4b}$$

This will lead us, in this case, to take for $\bar{\mathscr{H}}'$ the space $\mathscr{H}$ itself, following 3°) and (0.2), by identifying the element h of $\mathscr{H}$ with the element of $\bar{\mathscr{H}}'$, represented by the continuous linear form on $\bar{\mathscr{H}}$: $\bar{k} \mapsto (h \mid k)_{\mathscr{H}}$; we immediately obtain (0.2), with $h \in \bar{\mathscr{H}}' = \mathscr{H}$ and $\bar{k} \in \bar{\mathscr{H}} = \mathscr{H}'$:

(0.4c) $\quad \langle h, \bar{k} \rangle = (h \mid k)_{\mathscr{H}}, \qquad$ which is (0.4b) with h and k reversed.

The scalar product (0.4) of $h \in \mathscr{H}$ and $k \in \bar{\mathscr{H}}' = \mathscr{H}$ is nothing but their scalar product $(h \mid k)_{\mathscr{H}}$ in $\mathscr{H}$. The system of the 4 spaces $\mathscr{H}, \mathscr{H}', \bar{\mathscr{H}}, \bar{\mathscr{H}}'$ thus reduces to 2, $\mathscr{H} = \bar{\mathscr{H}}'$ and $\bar{\mathscr{H}} = \mathscr{H}'$.

If, moreover, we have an anti-involution on $\mathscr{H}$, the four spaces become canonically isomorphic. The isomorphism between $\mathscr{H}$ and $\mathscr{H}'$ is the following: we associate to $h \in \mathscr{H}$ the element of $\mathscr{H}'$ defining the continuous linear form on $\mathscr{H}$: $k \mapsto (k \mid \bar{h})$.

[(4)] The transpose of a continuous linear map u from E into F is a linear map ${}^t u$ from F' into E' defined by $\langle u(x), y' \rangle = \langle x, {}^t u(y') \rangle$ for $x \in E$ and $y' \in E'$; the transpose of an antilinear map u is an antilinear map ${}^t u$ defined by $\langle u(x), y' \rangle = \overline{\langle x, {}^t u(y') \rangle}$. The contragredient is always the inverse of the transpose.

[(5)] A sesquilinear form B on $E \times E$ is a complex-valued function such that the partial map $e \mapsto B(e, f)$ is linear and the partial map $f \mapsto B(e, f)$ is antilinear.

However, im analysis, it will in general be impossible to make these identifications: $\mathscr{H}$ will quite generally possess a natural anti-involution, allowing the identification of $\mathscr{H}$ and $\bar{\mathscr{H}}$, and hence also $\mathscr{H}'$ and $\bar{\mathscr{H}}'$, but the pairs $\mathscr{H}, \bar{\mathscr{H}}$ and $\bar{\mathscr{H}}', \mathscr{H}'$ will have to be distinguished despite being isomorphic*. Thus, in Example 2 on page 8, we will have $\bar{\mathscr{H}}^s = \mathscr{H}^s$ by the natural anti-involution of complex conjugation $f \mapsto \bar{f}$, but $(\mathscr{H}^s)' = (\bar{\mathscr{H}}^s)' = \mathscr{H}^{-s} \neq \mathscr{H}^s$.

Henceforth, we will speak of the conjugate space $\bar{E}$ of E without specifying which model was chosen. We will, however, always assume the identification of $\bar{E}'$ and $\overline{E'}$ following 3°) *and (0.2).*

Conjugate of a linear or antilinear map.

Let u be a continuous linear or antilinear map from E into F. Then $\bar{u}$ is a map *of the same nature*, from $\bar{E}$ into $\bar{F}$, defined by

$$\bar{u}\bar{e} = \overline{ue}, \qquad \forall e \in E. \tag{0.5}$$

If, in particular, $u = e' \in E'$, corresponding to $F = \mathbb{C}$, $\bar{u}$ is the element $\overline{e'}$ of $\overline{E'}$ defined by (0.2), given that we take $\mathbb{C}$ itself for $\overline{\mathbb{C}}$, and the usual conjugation $z \mapsto \bar{z}$.

The map $u \mapsto \bar{u}$ is an anti-isomorphism from $\mathscr{L}(E;F)$ (the space of continuous linear maps from E into F) onto $\mathscr{L}(\bar{E};\bar{F})$, so that we can consider the latter as the conjugate space of the former. We now have, for $e \in E$ and $f' \in F'$, the formula

$$\begin{aligned} \langle \overline{{}^t u} \bar{f}', \bar{e} \rangle &= \langle \overline{{}^t u f'}, \bar{e} \rangle \\ &= \overline{\langle {}^t u f', e \rangle} = \overline{\langle f', ue \rangle} \\ &= \langle \bar{f}', \overline{ue} \rangle = \langle \bar{f}', \bar{u}\bar{e} \rangle = \langle {}^t\bar{u}\bar{f}', \bar{e} \rangle \end{aligned} \tag{0.6}$$

so that

$$\overline{{}^t u} = {}^t\bar{u} \in \mathscr{L}(\bar{F}'; \bar{E}'). \tag{0.7}$$

This operator $\overline{{}^t u} = {}^t\bar{u}$ will also be denoted u^* and called *the adjoint of* u.

According to (0.6), we have

$$\langle u^* \bar{f}', \bar{e} \rangle = \overline{\langle f', ue \rangle} \quad \text{or} \quad \langle \bar{f}', \overline{ue} \rangle \tag{0.7b}$$

or, with (0.4):

$$(u^* \bar{f}' \mid e) = (\bar{f}' \mid ue). \tag{0.7c}$$

The reason for the use of letters such as $\bar{f}', e, ...$ is that they immediately indicate the space in which we work. But they sometimes have the drawback of drowning everything in the signs and indices. We may sometimes choose to represent (0.7b) and (0.7c) under the form

$$\left.\begin{aligned} \langle u^*\phi, \bar{\psi} \rangle &= \langle \phi, \overline{u\psi} \rangle \\ (u^*\phi \mid \psi) &= (\phi \mid u\psi) \end{aligned}\right\} \qquad \phi \in \bar{F}', \psi \in E. \tag{0.7d}$$

*"Dangerous corner". See N.Bourbaki, Élements de Mathémathiques, Fascicule I, Paris 1939. Page VI.

If E and F are Hilbert spaces and if we identify them with $\bar{E}'$ and $\bar{F}'$ by 4°), u^* becomes a continuous linear map from F to E, which is indeed the usual adjoint.

[It is the same if u is antilinear, except that the above equalities in (0.6) are to be replaced by

$$\text{(0.8)} \qquad \begin{aligned} ...\overline{\langle {}^t u f', e\rangle} &= \langle f', ue\rangle = \overline{\langle \bar{f}', \overline{ue}\rangle} \\ &= \overline{\langle \bar{f}', \bar{u}\bar{e}\rangle} = \overline{\langle {}^t\bar{u}\bar{f}', \bar{e}\rangle}. \end{aligned}$$

What replaces (0.7b) here is

$$\text{(0.9)} \qquad \langle u^*\bar{f}', \bar{e}\rangle = \langle f' = ue\rangle,$$

while (0.7c) is replaced by

$$\text{(0.10)} \qquad (u^*\bar{f}' \mid e) = (ue \mid \bar{f}').]$$

Finally, if E, F and G are three spaces, and if u maps E into F and v maps F into G, we have $\overline{v \circ u} = \bar{v} \circ \bar{u}$. If u is invertible, $\bar{u}$ is also invertible and $\bar{u}^{-1} = \overline{u^{-1}}$.

All these formulae are quite easy and quite mechanical. We have the following general rules:

1. To take the conjugate of an expression, we conjugate everything that is in the expression (examples: Equations (0.2) or (0.5)).

2. Above each letter, the parity of the number of bars is the same on either side of an $=$ sign; provided that we consider $*$ as equivalent to a bar ($u^* = \overline{{}^t u}$), that a letter placed after the vertical line in a scalar product ($|$) contains a bar ($(\alpha \mid \beta) = \langle \alpha, \bar{\beta}\rangle$) and that, if u is antilinear, ue contains a bar on e. This rule can easily be verified in (0.6) and (0.8).

§1. Hilbert subspaces of a topological vector space

Henceforth, E will always be a locally convex Hausdorff topological vector space over the field of complex numbers $\mathbb{C}$. A Hilbert (resp. pre-Hilbert) subspace $\mathscr{H}$ of E consists of a vector subspace $\mathscr{H}$ of E and a Hilbert (resp. pre-Hilbert) structure on $\mathscr{H}$ [this means, let us recall, a Hermitian form $(x, y) \mapsto (x \mid y)_{\mathscr{H}}$ that is linear in x and antilinear in y, which is required to be positive in the pre-Hilbert case, and in the Hilbert case, to be positive definite and such that $\mathscr{H}$ is complete with respect to the norm $x \mapsto \|x\|_{\mathscr{H}} = (x \mid x)^{1/2}$], for which the natural inclusion of $\mathscr{H}$ into E is continuous. This latter condition is equivalent to saying that, on $\mathscr{H}$, the topology defined by the norm is finer than the topology induced by E; and for that, it is necessary and sufficient that the unit ball of $\mathscr{H}$ is bounded in E. It should be noted that, following this definition, two distinct Hilbert structures on the same vector subspace of E are considered to define two distinct Hilbert subspaces of E.

Having given the definitions of Hilbert and pre-Hilbert subspaces without other hypotheses on E, *we will henceforth always assume, without explicit mention, that E is quasi-complete*[6] *with respect to its initial topology.* We will very

[6] A topological vector space is quasi-complete if all its closed *bounded* subsets are complete. For a metrisable space, quasi-completeness is equivalent to completeness.

frequently consider its weak topology $\sigma(E, E')$; naturally it is not necessarily quasi-complete with respect to this latter topology (except and only except if it is semi-reflexive[(7)]). On the contrary, it is quasi-complete with respect to the topology $\tau(E, E')$ of Mackey, because it is finer than the initial topology, and has a neighbourhood basis of the origin (the polar sets of the balanced convex weakly compact subsets of E') that is closed with respect to the initial topology[(8)]. It is easy to convince ourselves that the initial topology plays no role in the subsequent developments, and that we can simply consider a couple of dual spaces, E and E', such that E is τ-quasi-complete. We chose not to introduce this topology τ, which is often not well-known.

Proposition 0. *Let $\mathscr{H}$ be a vector subspace of E, equipped with a pre-Hilbert structure. If it is a pre-Hilbert subspace of E when we equip E with a coarser topology than the initial topology, but having the same bounded subsets (for example the weak topology $\sigma(E, E')$[(9)]), it is a pre-Hilbert subspace of E with respect to the initial topology. Let E'^* be the algebraic dual of E' (that is to say, the space of all linear forms on E'; so we have $E \subset E'^*$); let us equip it with the weak topology $\sigma(E'^*, E')$ (it is then a weakly complete subset of E). If $\mathscr{H}$ is a Hilbert subspace of E'^*, $\mathscr{H} \cap E$ (equipped with the pre-Hilbert structure induced by $\mathscr{H}$) is a Hilbert subspace of E: if $\mathscr{H}$ possesses a dense subspace contained in E, or a Hilbert basis contained in E, then $\mathscr{H}$ is a Hilbert subspace of E.*

Proof. The first claim is obvious, since $\mathscr{H}$ is a pre-Hilbert subspace of E if and only if the unit ball of $\mathscr{H}$ is bounded in E: this does not depend on the topology of E, but only on its bounded subsets.

Let $\mathscr{H}$ be a Hilbert subspace of E'^*. Then, let $h_n, n = 1, 2, ...$ be a Cauchy sequence in $\mathscr{H}$ contained in E; it converges to an element in $\mathscr{H}$ with respect to the topology of $\mathscr{H}$; but it is a Cauchy sequence in E, because $\mathscr{H} \cap E$, a weak pre-Hilbert subspace of E, is also a pre-Hilbert subspace of E with respect to its initial topology; so, since E is assumed to be quasi-complete with respect to its initial topology, the sequence also converges to an element in E with respect to the topology of E. Its limits in $\mathscr{H}$ and in E coincide, because they are its limit in E'^*. This indeed proves that $\mathscr{H} \cap E$ is complete, and hence that it is a Hilbert subspace of E. If $\mathscr{H}_0$ is a subspace of $\mathscr{H}$ contained in E, its closure in $\mathscr{H}$ is then in $\mathscr{H} \cap E$; so if $\mathscr{H}_0$ is dense in $\mathscr{H}$, $\mathscr{H}$ is in E. This is what we will obtain if $\mathscr{H}$ has a Hilbert basis in E, because the subspace which it generates is dense in $\mathscr{H}$ and contained in E. □

Example 1. On a finite-dimensional vector subspace of E, every positive definite Hermitian form defines a Hilbert structure whose topology is identical to

(7) For the weak topology $\sigma(E, E')$, see Bourbaki [1]. Chapter IV, §2, n° 1. As every bounded subset of E is always weakly pre-compact, saying that a weakly closed and bounded subset is weakly complete is equivalent to saying that it is weakly compact; and this is precisely the condition of semi-reflexivity of Mackey (Bourbaki [1], Chapter IV, §3, n°3, Theorem 1).

(8) For the topology τ of Mackey, see Bourbaki [1], Chapter IV, §2, n°3, Theorem 2 and Corollary. From this definition, it is immediate that the initial topology of E is coarser than $\tau(E, E')$. The fact that any space E which is quasi-complete with respect to its initial topology is also quasi-complete with respect to $\tau(E, E')$ then results from Bourbaki [1], Chapter I, §1, n°5, Proposition 8.

(9) Every weakly bounded subset is also bounded with respect to the initial topology (Mackey's Theorem, see Bourbaki [1], Chapter IV, §2, n°4, Theorem 3).

the topology induced by E, since they are both Hausdorff(10), so it defines a Hilbert subspace of E. For certain infinite-dimensional spaces E "monstrous" enough, all Hilbert subspaces are finite-dimensional.

For example, let E be an infinite-dimensional vector space, equipped with the finest locally convex topology(11). In this topology, every vector subspace is closed. If $\mathscr{H}$ is then a Hilbert subspace of E, all its vector subspaces are closed in E and thus in $\mathscr{H}$, so $\mathscr{H}$ is finite-dimensional.

Example 2. If E is the space $\mathscr{D}'$ of complex distributions on $\mathbb{R}^n$, $\mathscr{H} = L^2$ is a Hilbert subspace of E. Usually, we also write $\mathscr{H}^0 = L^2$, and define the space $\mathscr{H}^s$, with $s \geqq 0$ a natural number(12), as the space of functions (more precisely: classes of functions, up to Lebesgue equivalence) belonging to L^2, whose derivatives of order $\leqq s$, in the sense of distributions, are also functions in L^2 (usually, one writes H^s, but here we write $\mathscr{H}^s$). For every derivative index $p = (p_1, p_2, ..., p_n)$ of order $|p| = p_1 + p_2 + ... + p_n \leqq s$, one chooses a number $a_p > 0$; then we can equip $\mathscr{H}^s$ with a scalar product

$$(1.1) \qquad (f \mid g)_{\mathscr{H}^s} = \sum_{|p| \leqq s} \int_{\mathbb{R}^n} a_p D^p f D^p \bar{g} dx,$$

which is indeed a Hilbert subspace of $\mathscr{D}'$.

As a topological vector space, it is independent of the choice of a_p, but not as a Hilbert space.

We define $\mathscr{H}^{-s}$ as the dual of $\mathscr{H}^s$; equipped with its natural inclusion into $\mathscr{D}'$ obtained as the transpose of the natural dense inclusion of $\mathscr{D}$ into $\mathscr{H}^s$, and with a norm as the dual of $\mathscr{H}^s$, it is again a Hilbert subspace of $\mathscr{D}'$ (its Hilbert structure depends on the choice of a_p).

More generally, we define $\mathscr{H}^s$ for some real s by Fourier transformation, but we do not make use of it here. If now X is an open subset of $\mathbb{R}^d$, $\mathscr{H}^s(X)$, for a natural number $s \geqq 0$, is a subspace of $\mathscr{D}'(X)$ (space of distributions on X) consisting of functions in $L^2(X)$ whose derivatives of order $\leqq s$, in the sense of distributions, are in $L^2(X)$; we equip it with the scalar product in (1.1), where $\int_{\mathbb{R}^n}$ is replaced by $\int_X$, which depends on a_p and in fact is a Hilbert subspace of $\mathscr{D}'(X)$. $\mathscr{D}(X)$ is not dense in $\mathscr{H}^s(X)$ for $s \geqq 1$; let $\mathscr{H}_0^s$ be the closure of $\mathscr{D}(X)$ in $\mathscr{H}^s(X)$. Then we define $\mathscr{H}^{-s}(X)$ as the dual of $\mathscr{H}_0^s(X)$, equipped with the natural inclusion in $\mathscr{D}'(X)$ obtained by transposing the dense inclusion of $\mathscr{D}(X)$ into $\mathscr{H}_0^s(X)$; it is again a Hilbert subspace of $\mathscr{D}'(X)$.

Example 3. Let X be an oriented Riemann space of dimension n, of C^∞-class. Then there exists an operation $*$, which gives a correspondence between every differential form ω of degree p and a differential form $*\omega$ or ω^* of degree $n-p$; the operation $*$ is antilinear. Let $\mathscr{H}$ be the space of classes of measurable

(10) On a finite-dimensional vector space, there is only one possible Hausdorff vector space topology; see Bourbaki [1], Chapter I, §2, n°3, Theorem 2.

(11) On every vector space, there exists a locally convex topology which is finer than all other topologies, defined, for example, by the family of all semi-norms. Every semi-norm is continuous, hence so are all the linear forms, which means all the hyperplanes are closed, hence so are all vector subspaces, which are all intersections of hyperplanes.

(12) The spaces $\mathscr{H}^s$ are commonly used in the theory of partial differential equations. One can find a very complete study of it in, for example, Hörmander [1], Chapter II, Definition 2.4.1, with the notation $\mathscr{H}_{(s)}$.

differential forms, such that the positive form $\omega \wedge \omega^*$ of degree n is integrable; we can equip $\mathscr{H}$ with the scalar product:

$$(\alpha \mid \beta)_{\mathscr{H}} = \int_X \alpha \wedge \beta^*. \tag{1.2}$$

$\mathscr{H}$ is a Hilbert subspace of the space $\mathscr{D}'(X)$ of currents on X; it is used in the theory of harmonic forms(13). If X is compact, $\mathscr{H}$, as a topological vector space, does not depend on the Riemannian structure of X, that is to say, on the operation $*$ ($\mathscr{H}$ is the space of classes of measurable forms with L^2 coefficients on every compact subset of every chart of X); but its Hilbert structure does depend on it. If X is an open subset of $\mathbb{R}^n$, the subspace $\mathring{\mathscr{H}}$ of $\mathscr{H}$ of degree-0 forms is nothing but $L^2(X)$, with its usual norm.

Example 4. Let X be a locally compact topological space, and μ a non-negative Radon measure on X. For every $f^\bullet \in L^2(X,\mu)$ ($f^\bullet$ is a class of functions, not one function), we can define the Radon measure $f\mu$, where f is some function in the class $f^\bullet$. We will denote by $\Lambda^2(X,\mu)$ the space of these measures, equipped with the scalar product

$$\begin{aligned}(f\mu \mid g\mu)_{\Lambda^2(X,\mu)} &= (f^\bullet \mid g^\bullet)_{L^2(X,\mu)} \\ &= \int_X f\bar{g}d\mu.\end{aligned} \tag{1.2b}$$

$\Lambda^2(X,\mu)$ is a pre-Hilbert space; but as $f^\bullet \to f\mu$ is an isometry from L^2 onto Λ^2, it is Hilbert like L^2 itself. It is contained in the space $\mathscr{D}'^0(X)$ of Radon measures on X. In addition, if $f\mu$ converges weakly to 0 in Λ^2 (and a fortiori if it converges strongly), it converges weakly to 0 in $\mathscr{D}'^0$ because, for $\phi \in \mathscr{D}^0(X)$,

$$\langle f\mu, \phi\rangle = (f\mu \mid \bar{\phi}\mu)_{\Lambda^2}. \tag{1.2c}$$

So $\Lambda^2(X,\mu)$ is a Hilbert subspace of $\mathscr{D}'^0(X)$ with the weak topology, and also a Hilbert subspace of $\mathscr{D}'^0(X)$ with the strong topology, which has the same bounded subsets since $\mathscr{D}^0(X)$ is complete(14).

Let us remark that $\Lambda^2(X,\mu)$ is equipped with a natural anti-involution, the complex conjugation, induced by that of $\mathscr{D}'^0(X) : f\mu \mapsto \overline{f\mu} = \bar{f}\mu$. One can therefore, following 4°) on page 4, identify the four spaces $\Lambda^2, \bar{\Lambda}^2, (\Lambda^2)', \overline{(\Lambda^2)'}$; the element $f\mu$ of Λ^2 is, for example, identified with the element of $(\Lambda^2)'$ defining the continuous linear form on Λ^2: $g\mu \mapsto \int_X fgd\mu$. Equation (1.2c) can also be replaced by

$$\langle f\mu, \phi\rangle_{\mathscr{D}'^0,\mathscr{D}^0} = \langle f\mu, \phi\mu\rangle_{\Lambda^2,(\Lambda^2)'}. \tag{1.4d}$$

If X is an open subset of $\mathbb{R}^n$, and if μ is the Lebesgue measure dx, the function f, its class $f^\bullet$ and the measure or the distribution fdx are identified; then $L^2(X,dx)$ and $\Lambda^2(X,dx)$ are identified, as well as their duals and conjugate spaces.

(13) See de Rham [1], Chapter V, §24 and 25.

(14) $\mathscr{D}^0$ is the space of continuous functions with compact support, equipped with the usual topology of direct limits (Schwartz [1], Chapter III, §1). It is complete (loc. cit. Chapter III, Theorem 1) so the weakly bounded subsets of its dual $\mathscr{D}'^0$ are also strongly bounded (Bourbaki [1], Chapter IV, §3, n°2, Proposition 1).

Completion of a pre-Hilbert subspace of E

Let $\mathscr{H}_0$ be a pre-Hilbert subspace of E. The completion of $\mathscr{H}_0$ in E is any Hilbert subspace $\mathscr{H}$ of E, such that the pre-Hilbert structure of $\mathscr{H}_0$ is induced by $\mathscr{H}$, and in which $\mathscr{H}_0$ is dense.

Proposition 1. *Let $\mathscr{H}_0$ a pre-Hilbert subspace of E, and j its natural inclusion into E; let $\hat{\mathscr{H}}_0$ be the completion of $\mathscr{H}_0$*[15]*, and $\hat{\jmath}$ the continuous linear map from $\hat{\mathscr{H}}_0$ into E extending j. For $\mathscr{H}_0$ to have a completion $\mathscr{H}$ in E, it is necessary and sufficient for $\hat{\jmath}$ to be injective; or equivalently, it is necessary and sufficient for the unit ball B_0 of $\mathscr{H}_0$ to be closed in $\mathscr{H}_0$ with respect to the topology induced by E. In this case, $\mathscr{H}$ is unique; it is the image of $\hat{\mathscr{H}}_0$ under $\hat{\jmath}$, equipped with the Hilbert structure transported from $\hat{\mathscr{H}}_0$ under $\hat{\jmath}$.*

Proof. The complete space $\hat{\mathscr{H}}_0$ is in general not a subspace of E. But j, being continuous and linear, extends uniquely to a continuous linear map $\hat{\jmath}$ from $\hat{\mathscr{H}}_0$ into $\hat{E}$; in fact, as $\mathscr{H}_0$ is normed, its completion $\hat{\mathscr{H}}_0$ coincides with its quasi-completion $\hat{\mathscr{H}}_0$, so $\hat{\jmath}$ sends $\hat{\mathscr{H}}_0$ into the quasi-completion $\hat{E}$, which is E, since E is assumed to be quasi-complete. We have a commutative diagram:

$$\begin{array}{ccc} \mathscr{H}_0 & \xrightarrow[\subset]{\;j\;} & E \\ & \searrow \qquad \nearrow_{\hat{\jmath}} & \\ & \hat{\mathscr{H}}_0 & \end{array} \tag{1.3}$$

where $\mathscr{H}_0 \to \hat{\mathscr{H}}_0$ is the inclusion of $\mathscr{H}_0$ in its completion. The fact that j is injective in no way, in general, implies that $\hat{\jmath}$ is injective.

1°) Let us first suppose that there exists a completion $\mathscr{H}$ of $\mathscr{H}_0$ in E. Since $\mathscr{H}$ is complete, and $\mathscr{H}_0$ is dense in $\mathscr{H}$, there exists a canonical *isomorphism* of $\hat{\mathscr{H}}_0$ onto $\mathscr{H}$ and we have the following commutative diagram:

$$\begin{array}{ccccc} & & \overset{j}{\frown} & & \\ \mathscr{H}_0 & \xrightarrow[\subset]{} & \mathscr{H} & \xrightarrow[\subset]{} & E \\ & \searrow & \updownarrow & \nearrow_{\hat{\jmath}} & \\ & & \hat{\mathscr{H}}_0 & & \end{array} \tag{1.4}$$

Then $\hat{\jmath}$ is indeed injective, and $\mathscr{H}$ is indeed unique, since it is the image of $\hat{\jmath}(\hat{\mathscr{H}}_0)$, and its structure is transported by $\hat{\jmath}$ from that of $\hat{\mathscr{H}}_0$. The unit ball B_0 of $\mathscr{H}_0$ is the intersection $B \cap \mathscr{H}_0$ of the unit ball B of $\mathscr{H}$ with $\mathscr{H}_0$; B is weakly compact in $\mathscr{H}$, so weakly compact in E, and so closed in E; hence, B_0 is indeed closed in $\mathscr{H}_0$ with respect to the topology induced by E.

2°) Let us now suppose that $\hat{\jmath}$ is injective. Then, if we denote the image $\hat{\jmath}(\hat{\mathscr{H}}_0)$ by $\mathscr{H}$, equipped with the Hilbert structure carried over by $\hat{\jmath}$ from that of

[15] It is perhaps ambiguous to say “the” completion. There are several possible definitions of it; all completions are canonically isomorphic in every way. $\hat{\mathscr{H}}_0$ thus denotes *any* Hilbert space in which $\mathscr{H}_0$ is a dense vector space, with the induced pre-Hilbert structure.

$\hat{\mathscr{H}}_0$, then $\mathscr{H}$ is Hilbert and the inclusion $\mathscr{H} \to E$ is continuous since $\hat{\jmath}$ is continuous, so $\mathscr{H}$ is a Hilbert subspace of E; as $\mathscr{H}_0$ is dense in $\hat{\mathscr{H}}_0$, it is dense in $\mathscr{H}$, and $\mathscr{H}$ is the completion of $\mathscr{H}_0$ in E.

3°) Lastly, let us suppose that the unit ball B_0 of $\mathscr{H}_0$ is closed in $\mathscr{H}_0$ with respect to the topology induced by E. Let h be an element of $\hat{\mathscr{H}}_0$ such that $\hat{\jmath}(h) = 0$. There exists a sequence of elements h_n of $\mathscr{H}_0$ which, as $n \to \infty$, converges to h in $\hat{\mathscr{H}}_0$; $\jmath(h_n) = h_n$, so h_n converge to $\hat{\jmath}(h) = 0$ in E. Let $\epsilon > 0$; there exists N such that, for $m \geqq N$ and $n \geqq N$, $\|h_m - h_n\|_{\mathscr{H}_0} \leqq \epsilon$; as $n \to \infty$, $h_m - h_n$ converge to h_m in E, and as the ball of radius ϵ of $\mathscr{H}_0$ is closed with respect to the topology induced by E, we have $\|h_m\|_{\mathscr{H}_0} \leqq \epsilon$ for $m \geqq N$. This proves that h_m converges to 0 in $\mathscr{H}_0$ as $m \to \infty$, so $h = 0$, and $\hat{\jmath}$ is injective; we have arrived at 2°), and Proposition 1 is shown.

□

Counterexample. Let $E = L^2(\mathbb{R})$, and let $\mathscr{H}_0$ be a subspace of *continuous* functions of L^2, equipped with the scalar product

$$(f \mid g)_{\mathscr{H}_0} = \int_{\mathbb{R}} f(x)\overline{g(x)}dx + f(0)\overline{g(0)}. \tag{1.5}$$

$\mathscr{H}_0$ is a pre-Hilbert subspace of E.

We can identify $\hat{\mathscr{H}}_0$ with the product $L^2 \times \mathbb{C}$, equipped with the scalar product

$$((f,\alpha) \mid (g,\beta))_{\hat{\mathscr{H}}_0} = \int_{\mathbb{R}} f(x)\overline{g(x)}dx + \alpha\bar{\beta}, \tag{1.6}$$

with the inclusion from $\mathscr{H}_0$ into $\hat{\mathscr{H}}_0$ defined by $f \mapsto (f, f(0))$.

Then the map $\hat{\jmath}$ from $\hat{\mathscr{H}}_0$ into E is defined by $(f,\alpha) \mapsto f$, the only continuous linear map from $\hat{\mathscr{H}}_0$ into E which, composed with the inclusion of $\mathscr{H}_0$ in $\hat{\mathscr{H}}_0$, gives the inclusion of $\mathscr{H}_0$ in E: $f \mapsto (f, f(0)) \mapsto f$. Then $\hat{\jmath}$ is not injective; its kernel is $\{0\} \times \mathbb{C}$. It can be seen here that B_0 is not closed in $\mathscr{H}_0$ with respect to the topology induced by E. This is because the closure $\bar{B}_0$ of B_0 in E is the unit ball of L^2 (any $f \in L^2$, of norm at most 1, is the limit in L^2 of a sequence $f_n \in L^2$ of continuous functions satisfying $\|f_n\|_{L^2} \leqq 1$ and $f_n(0) = 0$, so belonging to B_0); then $\overline{B_0} \cap \mathscr{H}_0$ is strictly larger than B_0, and contains, for example, all continuous functions $f \in L^2$ satisfying $\|f\|_{L^2} \leqq 1$ and $|f(0)| > 1$, which do not belong to B_0.

Q-completion in E of a pre-Hilbert subspace $\mathscr{H}_0$ of E

Let us suppose that the conditions stated in Proposition 1 do not hold. Then $\hat{\jmath}$ is not injective; let $\mathscr{N} = (\hat{\jmath})^{-1}(\{0\})$ be its kernel; we certainly have $\mathscr{N} \cap \mathcal{H}_0 = \{0\}$. $\hat{\mathscr{H}}_0/\mathscr{N}$ is a Hilbert space, and $\hat{\jmath}$ factorises into $\hat{\mathscr{H}}_0 \xrightarrow{\pi} \hat{\mathscr{H}}_0/\mathscr{N} \xrightarrow{\hat{\jmath}^\bullet} E$, where π is the canonical map from $\hat{\mathscr{H}}_0$ onto the quotient $\hat{\mathscr{H}}_0/\mathscr{N}$, and where $\hat{\jmath}^\bullet$ is injective. We can then transport the Hilbert structure of $\hat{\mathscr{H}}_0/\mathscr{N}$ onto $\mathscr{H} = \hat{\jmath}(\hat{\mathscr{H}}_0)$ via $\hat{\jmath}^\bullet$; $\mathscr{H}$ is a Hilbert subspace of E, in which $\mathcal{H}_0$ is dense. But it induces a weaker norm on $\mathscr{H}_0$ than that of $\mathscr{H}_0$, namely that of $\hat{\mathscr{H}_0}/\mathscr{N}$; we will say that *$\mathscr{H}$ is the Q-completion of $\mathscr{H}_0$ in E*, Q standing for quotient. In the preceding example, the Q-completion of $\mathscr{H}_0$ in L^2 is L^2. We can characterise $\mathscr{H}$ as follows:

Proposition 1b. *The Q-completion $\mathscr{H}$ of $\mathscr{H}_0$ in E is the smallest Hilbert subspace of E containing $\mathscr{H}_0$, and its norm is the largest Hilbert subspace norm of E which induces a weaker norm on $\mathscr{H}_0$ than that of $\mathscr{H}_0$ (that is to say, such that the inclusion of $\mathscr{H}_0$ in $\mathscr{H}$ is of norm $\leqq 1$).*

Proof. Let $\mathscr{K}$ be a Hilbert subspace of E containing $\mathscr{H}_0$. Then the inclusion j of $\mathscr{H}_0$ in E factorises into $\mathscr{H}_0 \to \mathscr{K} \to E$, composed of continuous inclusions. By continuous extensions, $\hat{\jmath}$ factorises into $\hat{\mathscr{H}_0} \to \mathscr{K} \to E$, so $\hat{\jmath}(\hat{\mathscr{H}_0}) = \mathscr{H} \subset \mathscr{K}$; $\mathscr{H}$ is thus the smallest Hilbert subspace of E containing $\mathscr{H}_0$. If, moreover, $\mathscr{H}_0 \to \mathscr{K}$ has norm $\leqq 1$, then $\hat{\mathscr{H}_0} \to \mathscr{K}$ has norm $\leqq 1$, so, by passing to quotients, $\hat{\mathscr{H}_0}/\mathscr{N} \to \mathscr{K}$ has norm $\leqq 1$, hence so does $\mathscr{H} \to \mathscr{K}$; the norm induced by $\mathscr{K}$ on $\mathscr{H}$ is smaller than that of $\mathscr{H}$. □

§2. The set Hilb(E) of Hilbert subspaces of E

We shall denote by Hilb(E) the set of Hilbert subspaces of E. It possesses a remarkable structure.

1°) There exists a multiplication law of elements of Hilb(E) by non-negative real numbers.

Let $\mathscr{H}$ be a Hilbert subspace of E, λ a non-negative real number. We denote by $\lambda\mathscr{H}$ the space $\{0\}$ if $\lambda = 0$, and if $\lambda > 0$, the space $\mathscr{H}$ the space $\mathscr{H}$ equipped with a new scalar product, obtained by multiplying the old scalar product by $\frac{1}{\lambda}$ (the following relations will show the necessity of this choice):

$$(2.1) \qquad (h \mid k)_{\lambda\mathscr{H}} = \frac{1}{\lambda}(h \mid k)_{\mathscr{H}}.$$

We thus have

$$(2.2) \qquad \|h\|_{\lambda\mathscr{H}} = \frac{1}{\sqrt{\lambda}}\|h\|_{\mathscr{H}}.$$

We obviously have associativity

$$(2.3) \qquad (\lambda\mu)\mathscr{H} = \lambda(\mu\mathscr{H}), \qquad \lambda \geqq 0, \mu \geqq 0.$$

2°) There exists an addition law on Hilb(E).

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert subspaces of E; we will define a Hilbert subspace $\mathscr{H}_1+\mathscr{H}_2$ (beware that we do not necessarily have $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$).

Firstly, let us consider the Hilbert sum $\mathscr{H}_1 \oplus \mathscr{H}_2$; it is the product $\mathscr{H}_1 \times \mathscr{H}_2$ equipped with the scalar product

$$(2.4) \qquad \big((h_1, h_2) \mid (k_1, k_2)\big)_{\mathscr{H}_1 \oplus \mathscr{H}_2} = \big(h_1 \mid k_1\big)_{\mathscr{H}_1} + \big(h_2 \mid k_2\big)_{\mathscr{H}_2}.$$

Then we also have

$$(2.5) \qquad \|h_1, h_2\|^2_{\mathscr{H}_1 \oplus \mathscr{H}_2} = \|h_1\|^2_{\mathscr{H}_1} + \|h_2\|^2_{\mathscr{H}_2}.$$

Of course, this is not a subspace of E. But there exists a natural map Φ from $\mathscr{H}_1 \oplus \mathscr{H}_2$ into E, defined by

$$\Phi(h_1, h_2) = h_1 + h_2. \tag{2.6}$$

Φ is linear and continuous. Its kernel $\mathscr{N}$ is the set of pairs (h_1, h_2) such that $h_1 + h_2 = 0$, and its image is the sum $\mathscr{H}_1 + \mathscr{H}_2$. The kernel $\mathscr{N}$ is closed since Φ is continuous, so $(\mathscr{H}_1 \oplus \mathscr{H}_2)/\mathscr{N}$ is again a Hilbert space, and we have a commutative diagram:

$$\begin{array}{ccc} \mathscr{H}_1 \oplus \mathscr{H}_2 & \xrightarrow{\quad\Phi\quad} & E \\ & {\scriptstyle\pi}\searrow \qquad \nearrow{\scriptstyle\Phi^\bullet} & \\ & (\mathscr{H}_1 \oplus \mathscr{H}_2)/\mathscr{N} & \end{array} \tag{2.7}$$

where π is the canonical map from $\mathscr{H}_1 \oplus \mathscr{H}_2$ onto the quotient $(\mathscr{H}_1 \oplus \mathscr{H}_2)/\mathscr{N}$, and where $\Phi^\bullet$ is continuous, linear and injective. If $\mathscr{K}$ is orthogonal to $\mathscr{N}$ in $\mathscr{H}_1 \oplus \mathscr{H}_2$, π is an isomorphism (with respect to the Hilbert structure) from $\mathscr{K}$ onto $(\mathscr{H}_1 \oplus \mathscr{H}_2)/\mathscr{N}$.

The image of Φ, equipped with the Hilbert structure carried over by $\Phi^\bullet$ from that of $(\mathscr{H}_1 \oplus \mathscr{H}_2)/\mathscr{N}$, will then be called the sum space, and will be denoted by $\mathscr{H}_1 + \mathscr{H}_2$. $\mathscr{H}_1 + \mathscr{H}_2$ is a Hilbert space, and its inclusion in E is continuous since $\Phi^\bullet$ is continuous; it is thus a Hilbert subspace of E. Moreover, Φ is an isomorphism (with respect to the Hilbert structure) from $\mathscr{K}$ onto $\mathscr{H}_1 + \mathscr{H}_2$.

For an element $h^\bullet$ of $(\mathscr{H}_1 \oplus \mathscr{H}_2)/\mathscr{N}$, the quotient norm is defined by

$$\begin{aligned} \|h^\bullet\|^2_{(\mathscr{H}_1 \oplus \mathscr{H}_2)/\mathscr{N}} &= \inf_{\pi(h_1,h_2)=h^\bullet} \left\|(h_1, h_2)\right\|^2_{\mathscr{H}_1 \oplus \mathscr{H}_2} \\ &= \inf_{\pi(h_1,h_2)=h^\bullet} \left(\|h_1\|^2_{\mathscr{H}_1} + \|h_2\|^2_{\mathscr{H}_2}\right). \end{aligned} \tag{2.8}$$

From this, we deduce the following formula for an element h of $\mathscr{H}_1 + \mathscr{H}_2$:

$$\|h\|^2_{\mathscr{H}_1 + \mathscr{H}_2} = \inf_{h_1 + h_2 = h} \left(\|h_1\|^2_{\mathscr{H}_1} + \|h_2\|^2_{\mathscr{H}_2}\right). \tag{2.9}$$

Let us note that, in (2.8) and (2.9), the infimum is in fact a minimum, and we have exactly

$$\|h\|^2_{\mathscr{H}_1 + \mathscr{H}_2} = \|h_1\|^2_{\mathscr{H}_1} + \|h_2\|^2_{\mathscr{H}_2}, \tag{2.10}$$

for the *unique* element (h_1, h_2) of $\mathscr{H}_1 \oplus \mathscr{H}_2$ belonging to the orthogonal $\mathscr{K}$ of the kernel $\mathscr{N}$ and such that $h_1 + h_2 = h$.

Moreover, for h and k in $\mathscr{H}_1 + \mathscr{H}_2$, we have

$$(h \mid k)_{\mathscr{H}_1 + \mathscr{H}_2} = (h_1 \mid k_1)_{\mathscr{H}_1} + (h_2 \mid k_2)_{\mathscr{H}_2}, \tag{2.11}$$

with $h_1 + h_2 = h$, $k_1 + k_2 = k$, *provided that at least one of the elements* (h_1, h_2), (k_1, k_2) *of* $\mathscr{H}_1 \oplus \mathscr{H}_2$ *is orthogonal to* $\mathscr{N}$.

We can thus define $\mathscr{H}_1 + \mathscr{H}_2$ without going via $\mathscr{H}_1 \oplus \mathscr{H}_2$ and Φ, by simply saying that, as a vector space, it is the sum of subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ of

E, and that its norm is given by (2.9); but it is the method above that shows that this norm is Hilbert (because a quotient $(\mathscr{H}_1 \oplus \mathscr{H}_2)/\mathscr{N}$ of a Hilbert space by a closed subspace is again a Hilbert space).

If $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$, $\mathscr{N}$ is reduced to $\{0\}$; for $h \in \mathscr{H}_1 + \mathscr{H}_2$, there is a unique pair (h_1, h_2) of $\mathscr{H}_1 \oplus \mathscr{H}_2$ such that $h_1 + h_2 = h$, and Φ is an isomorphism of $\mathscr{H}_1 \oplus \mathscr{H}_2$ onto $\mathscr{H}_1 + \mathscr{H}_2$. $\mathscr{H}_1$ and $\mathscr{H}_2$ are two closed orthogonal subspaces and their Hilbert structures are induced by that of $\mathscr{H}_1 + \mathscr{H}_2$.

We have the formulae

$$\begin{cases} \lambda\,(\mathscr{H}_1 + \mathscr{H}_2) = \lambda\mathscr{H}_1 + \lambda\mathscr{H}_2, & \lambda \geqq 0, \\ (\lambda + \mu)\,\mathscr{H} = \lambda\mathscr{H} + \mu\mathscr{H}, & \lambda, \mu \geqq 0. \end{cases} \tag{2.11b}$$

The first formula is trivial; let us show the second, which is less so. The two sides coincide as vector spaces, so it remains to show that their norms coincide. Let us consider the case where $\lambda > 0$ and $\mu > 0$, since otherwise the result is trivial.

Let $h \in \mathscr{H}$. Let us find its norm in $\lambda\mathscr{H} + \mu\mathscr{H}$. Let h_1, h_2 be two elements in $\mathscr{H}$ such that $h_1 + h_2 = h$; under what condition is (h_1, h_2) in $\mathscr{K}$, the space orthogonal to $\mathscr{N}$ in $\lambda\mathscr{H} \oplus \mu\mathscr{H}$? We must have

$$(h_1 \mid k_1)_{\lambda\mathscr{H}} + (h_2 \mid k_2)_{\mu\mathscr{H}} = 0, \text{ or} \tag{2.11c}$$

$$\frac{1}{\lambda}(h_1 \mid k_1)_{\mathscr{H}} + \frac{1}{\mu}(h_2 \mid k_2)_{\mathscr{H}} = 0, \tag{2.12}$$

for every pair (k_1, k_2) in $\mathscr{H} \times \mathscr{H}$ such that $k_1 + k_2 = 0$. This is equivalent to

$$\left(\frac{h_1}{\lambda} - \frac{h_2}{\mu} \mid k_1\right)_{\mathscr{H}} = 0 \tag{2.13}$$

for every k_1 in $\mathscr{H}$, which in turn is equivalent to $\frac{h_1}{\lambda} - \frac{h_2}{\mu} = 0$.

The two equations

$$\begin{cases} h_1 + h_2 = h \\ \frac{h_1}{\lambda} - \frac{h_2}{\mu} = 0 \end{cases} \tag{2.14}$$

give

$$h_1 = \frac{\lambda}{\lambda + \mu} h, \qquad h_2 = \frac{\mu}{\lambda + \mu} h. \tag{2.15}$$

Then

$$\begin{aligned} \|h\|^2_{\lambda\mathscr{H}+\mu\mathscr{H}} &= \|h_1\|^2_{\lambda\mathscr{H}} + \|h_2\|^2_{\mu\mathscr{H}} \\ &= \frac{\lambda^2}{(\lambda+\mu)^2}\,\|h\|^2_{\lambda\mathscr{H}} + \frac{\mu^2}{(\lambda+\mu)^2}\,\|h\|^2_{\mu\mathscr{H}} \\ &= \frac{\lambda}{(\lambda+\mu)^2}\,\|h\|^2_{\mathscr{H}} + \frac{\mu}{(\lambda+\mu)^2}\,\|h\|^2_{\mathscr{H}} \\ &= \frac{1}{\lambda+\mu}\,\|h\|^2_{\mathscr{H}} \\ &= \|h\|^2_{(\lambda+\mu)\mathscr{H}}\,, \end{aligned} \tag{2.16}$$

which shows the second formula in (2.11b).

The addition law in Hilb(E) is trivially commutative, and has the Hilbert subspace $\{0\}$ as its identity element. Moreover, it is associative:

$$(\mathscr{H}_1 + \mathscr{H}_2) + \mathscr{H}_3 = \mathscr{H}_1 + (\mathscr{H}_2 + \mathscr{H}_3). \tag{2.17}$$

Each side can be written as $\mathscr{H}_1 + \mathscr{H}_2 + \mathscr{H}_3$, and can be defined directly as the sum subspace, equipped with the norm

$$\|h\|^2_{\mathscr{H}_1+\mathscr{H}_2+\mathscr{H}_3} = \inf_{h_1+h_2+h_3=h} \left(\|h_1\|^2_{\mathscr{H}_1} + \|h_2\|^2_{\mathscr{H}_2} + \|h_3\|^2_{\mathscr{H}_3} \right), \tag{2.18}$$

where the proof of the Hilbert subspace property of $\mathscr{H}_1 + \mathscr{H}_2 + \mathscr{H}_3$ uses the natural map Φ from $\mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3$ into E, defined by $\Phi(h_1, h_2, h_3) = h_1 + h_2 + h_3$.

3°) There exists an order relation in Hilb(E).

Let $\mathscr{H}_1, \mathscr{H}_2$ be two Hilbert subspaces of E. We will write $\mathscr{H}_1 \leqq \mathscr{H}_2$, if $\mathscr{H}_1 \subset \mathscr{H}_2$ and if the inclusion of $\mathscr{H}_1$ in $\mathscr{H}_2$ is continuous with norm $\leqq 1$, which translates to

$$\|h\|_{\mathscr{H}_1} \geqq \|h\|_{\mathscr{H}_2} \qquad \text{for } h \in \mathscr{H}_1. \tag{2.19}$$

This is also equivalent to saying that the open (resp. closed) unit ball of $\mathscr{H}_1$ is contained in the open (resp. closed) unit ball of $\mathscr{H}_2$.

For example, if we return to Example 2 on p.8, we have $\mathscr{H}^s(X) \geqq \mathscr{H}^t(X)$ for $s \leqq t$, if the collection of a_p, $|p| \leqq s$ extends to the collection of a_p, $|p| \leqq t$.

It is worth noting that the inequality $\leqq$ translates to the inclusion $\subset$, but to the inverse relation $\geqq$ between the norms. It is certainly an order relation because, if $\mathscr{H}_1 \leqq \mathscr{H}_2$ and $\mathscr{H}_2 \leqq \mathscr{H}_1$, the two vector spaces are the same with the same norm, hence the same scalar product, according to the well-known relation

$$4(h \mid k)_{\mathscr{H}} = \|h+k\|^2_{\mathscr{H}} - \|h-k\|^2_{\mathscr{H}} + i\,\|h+ik\|^2_{\mathscr{H}} - i\,\|h-ik\|^2_{\mathscr{H}}. \tag{2.19b}$$

For $\mathscr{H} \neq \{0\}$, we trivially have

$$\lambda\mathscr{H} \leqq \mathscr{H} \qquad \Longleftrightarrow \qquad \lambda \leqq 1. \tag{2.20}$$

Moreover, $\mathscr{H}_1 + \mathscr{H}_2 \geqq \mathscr{H}_1$ and $\geqq \mathscr{H}_2$. Indeed, on the one hand, $\mathscr{H}_1 + \mathscr{H}_2 \supset \mathscr{H}_1$, and on the other hand, for $h \in \mathscr{H}_1$, we have

$$\begin{aligned} \|h\|^2_{\mathscr{H}_1} &= \left\|(h,0)\right\|^2_{\mathscr{H}_1 \oplus \mathscr{H}_2} \\ &\geqq \inf_{h_1+h_2=h} \left\|(h_1,h_2)\right\|^2_{\mathscr{H}_1 \oplus \mathscr{H}_2} = \|h\|^2_{\mathscr{H}_1+\mathscr{H}_2}. \end{aligned} \tag{2.21}$$

Proposition 2. *Let $\mathscr{H}_1, \mathscr{H}_2$ be two Hilbert subspaces of E. For $\mathscr{H}_1 \subset \mathscr{H}_2$ to hold, it is necessary and sufficient that there exists a constant $c \geqq 0$ such that $\mathscr{H}_1 \leqq c\mathscr{H}_2$.*

Proof. The condition is trivially sufficient, so let us show that it is necessary. Let $\mathscr{H}_1 \subset \mathscr{H}_2$. The inclusion of $\mathscr{H}_1$ in $\mathscr{H}_2$ is continuous according to the closed graph theorem[16] (the graph of this inclusion is the diagonal of $\mathscr{H}_1 \times \mathscr{H}_1$; as the inclusion of $\mathscr{H}_1$ in E is continuous, it is closed in $\mathscr{H}_1 \times E$, hence closed in $\mathscr{H}_1 \times \mathscr{H}_2$). Let $\sqrt{c}$ be its norm; then we have, for $h \in \mathscr{H}_1$,

$$\|h\|_{c\mathscr{H}_2} = \frac{1}{\sqrt{c}}\,\|h\|_{\mathscr{H}_2} \leqq \|h\|_{\mathscr{H}_1}\,, \tag{2.22}$$

which proves the proposition. □

Corollary. *For $\mathscr{H}_1$ and $\mathscr{H}_2$ to be the same vector subspace of E (with potentially different Hilbert structures), it is necessary and sufficient that there exists a constant $c \geqq 0$ such that*

$$\mathscr{H}_1 \leqq c\mathscr{H}_2, \qquad \mathscr{H}_2 \leq c\mathscr{H}_1. \tag{2.23}$$

We then say that $\mathscr{H}_1$ and $\mathscr{H}_2$ are equivalent, $\mathscr{H}_1 \sim \mathscr{H}_2$.

Let Γ be a salient convex cone (that is to say, such that $\Gamma \cap (-\Gamma) = \{0\}$) in a vector space F. Then there exists a structure on Γ analogous to the one we just defined on $\mathrm{Hilb}(E)$: the multiplication by non-negative scalars and addition are induced by those of F; moreover, Γ defines an order structure on F, and so a fortiori on Γ, by

$$u \leqq v \qquad \Longleftrightarrow \qquad v - u \in \Gamma. \tag{2.24}$$

This order structure on Γ possesses a fundamental property with respect to addition: *for $u \leqq v$, it is necessary and sufficient that there exists $w \in \Gamma$ such that $v = u + w$, and this w is unique (it is $w = v - u$).*

We will prove (Theorem 1) that $\mathrm{Hilb}(E)$, with respect to the three structures which we endowed on it, is isomorphic to a convex salient cone Γ, in a vector space F; from this, we will have (Proposition 14) that: *for $\mathscr{H} \geqq \mathscr{H}_1$, it is necessary and sufficient that there exists $\mathscr{H}_2 \in \mathrm{Hilb}(E)$ such that $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$, and such a Hilbert subspace $\mathscr{H}_2$ is unique.*

Proposition 3. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert subspaces of E. For $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$, it is necessary and sufficient that $\mathscr{H}_1$ and $\mathscr{H}_2$ are alien with respect to the order relation, that is to say, that $\mathscr{K} \leqq \mathscr{H}_1$ and $\mathscr{K} \leqq \mathscr{H}_2$ implies $\mathscr{K} = \{0\}$.*

Proof. The condition is trivially necessary, so let us show that it is sufficient. Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are alien. Let us define the following scalar product on $\mathscr{H}_1 \cap \mathscr{H}_2$:

$$(h \mid k) = (h \mid k)_{\mathscr{H}_1} + (h \mid k)_{\mathscr{H}_2}. \tag{2.25}$$

This defines on $\mathscr{H}_1 \cap \mathscr{H}_2$ the structure of a pre-Hilbert subspace of E. But it is complete; since, if $(h_n), n = 1, 2, ...$ is a Cauchy sequence, it is a fortiori a Cauchy sequence in $\mathscr{H}_1$ and in $\mathscr{H}_2$, so it converges to a limit in each of these two spaces; these two limits coincide, because the convergence in $\mathscr{H}_1$ or in $\mathscr{H}_2$ implies the convergence in E; the limits are thus equal to an element h in $\mathscr{H}_1 \cap \mathscr{H}_2$, and h_n converges to h in $\mathscr{H}_1 \cap \mathscr{H}_2$. So $\mathscr{H}_1 \cap \mathscr{H}_2$ is a Hilbert subspace of E; it satisfies $\mathscr{H}_1 \cap \mathscr{H}_2 \leqq \mathscr{H}_1$ and $\mathscr{H}_1 \cap \mathscr{H}_2 \leqq \mathscr{H}_2$, so $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$, which proves the Proposition. □

(16) Bourbaki [1], Chapter I, §3, n°3, Corollary 5 of Theorem 1.

Remark. The three structures defined on Hilb(E) can be extended to the set of pre-Hilbert subspaces of E. Then *the completion, if it exists, or the Q-completion of a pre-Hilbert subspace $\mathscr{H}_0$ of E (Proposition 1b), is the smallest Hilbert subspace of E which is larger than $\mathscr{H}_0$, with respect to the order relation $\leqq$.*

§3. Kernels relative to E

Let E' be the dual of E. We define a kernel relative to E as a linear map from $\bar{E}'$ into E, continuous with respect to the weak topologies $\sigma(\bar{E}', \bar{E})$ and $\sigma(E, E')$. We know that it is then a fortiori continuous with respect to the topology of $\bar{E}'$ and the initial topology of E[(17)].

Let H be a kernel of E. For $\bar{e}' \in \bar{E}'$, $H\bar{e}'$ is an element of E; thus, for $f' \in E'$, we can define the scalar product $\langle H\bar{e}', f'\rangle$; it will play an essential role throughout. We can also write it as $(H\bar{e}' \mid \bar{f}')$ following (0.4). The adjoint $H^* = \overline{{}^tH} = {}^t\bar{H}$ also maps $\bar{E}'$ into E, and it is again a kernel (so $\mathscr{L}(\bar{E}; E)$ is equipped with a natural anti-involution); it is defined by (0.7b) or (0.7c):

$$\begin{cases} \forall e' \in E', \forall f' \in E', \langle H^*\overline{e'}, f'\rangle = \overline{\langle H\bar{f}', e'\rangle}, \\ \text{or } (H^*\bar{e}' \mid \bar{f}') = (\bar{e}' \mid H\bar{f}'). \end{cases} \tag{3.1}$$

We say that a kernel H is Hermitian if $H^* = H$; this translates to

$$\begin{gathered} \langle H\bar{e}, f'\rangle = \overline{\langle H\bar{f}', e'\rangle}, \qquad e' \in E', f' \in E', \text{ or} \\ (H\bar{e}' \mid \bar{f}') = (\bar{e}' \mid H\bar{f}'); \end{gathered} \tag{3.2}$$

which is equivalent, as it is well-known, to

$$\forall e' \in E', \langle H\bar{e}', e'\rangle \text{ or } (H\bar{e}' \mid \bar{e}') \text{ is real.} \tag{3.3}$$

Finally, H is said to be non-negative if

$$\forall e' \in E', \langle H\bar{e}', e'\rangle \geqq 0; \tag{3.4}$$

a non-negative kernel is Hermitian.

Proposition 4. *Every Hermitian linear map from $\bar{E}'$ into E is a kernel (in other words, it is weakly continuous).*

Proof. If $\bar{e}'$ converges to 0 with respect to $\sigma(\bar{E}', \bar{E})$, and so e' to 0 with respect to $\sigma(E', E)$, $\overline{\langle H\bar{f}', e'\rangle}$ converges to 0 for all fixed f'. Since H is Hermitian, $\langle H\bar{e}', f'\rangle$ also converges to 0, and $H\bar{e}'$ converges to 0 with respect to $\sigma(E, E')$. □

Instead of considering linear maps from $\bar{E}'$ into E, we can consider sesquilinear forms on $E' \times E'$ (or on $\bar{E}' \times \bar{E}'$). A kernel H defines the form $\tilde{H}$

$$\tilde{H}(e', f') = \langle H\bar{f}', e'\rangle. \tag{3.5}$$

[(17)]If u is continuous with respect to $\sigma(\bar{E}', \bar{E})$ and $\sigma(E, E')$, it is a fortiori continuous with respect to $\sigma(\bar{E}', \bar{E})$ and $\sigma(E'', E')$ so, according to Bourbaki [1], Chapter IV, §4, n°2, Proposition 6, it is continuous with respect to the strong topology of $\bar{E}'$ and the strong topology of E'', so a fortiori continuous with respect to the strong topology of $\bar{E}$ and the initial topology of E, the initial topology of E being coarser than the topology induced by the strong topology of E''.

Proposition 5. *For a sesquilinear form on $E' \times E'$ to be defined by a kernel according to (3.5), it is necessary and sufficient for it to be separately weakly continuous, and this kernel is then unique. The form is Hermitian (resp. non-negative), if and only if this kernel is Hermitian (resp. non-negative).*

Proof. 1) Let H be a kernel. If e' converges weakly to 0, then, for a fixed f', the right-hand side of (3.5) converges to 0. If f' converges weakly to 0, $H\bar{f}'$ converges to 0 in E by the weak continuity of H, so $\langle H\bar{f}', e'\rangle$ converges again to 0 for fixed e'.

2) Let B be a separately weakly continuous sesquilinear form on $E' \times E'$. For a fixed $\bar{f}'$ in $\bar{E}'$, $e' \mapsto B(e', f')$ is a weakly continuous form on E', so there exists a unique element $H\bar{f}'$ of E such that

$$B(e', f') = \langle H\bar{f}', e'\rangle. \tag{3.6}$$

H is a linear map from $\bar{E}'$ into E. If $\bar{f}'$ converges weakly to 0 in E', $B(e', f')$ converges to 0 for every fixed e', so $H\bar{f}'$ converges weakly to 0: H is weakly continuous, so it is a kernel. The final statement is obvious. □

Remark 1. More generally, the form $\tilde{H}^*$ associated to the adjoint H^* of H is the Hermitian transpose of $\tilde{H}$:

$$\tilde{H}(e', f') = \overline{\tilde{H}(f', e')}. \tag{3.6b}$$

Remark 2. Let E'^* be the algebraic dual of E'. Every linear form on E' is continuous with respect to the topology $\sigma(E', E'^*)$, so every sesquilinear form on $E' \times E'$ is separately continuous; but E' is the dual of E'^*, and its corresponding weak topology is $\sigma(E', E'^*)$; so *every sesquilinear form on $E' \times E'$ comes from a kernel H relative to E'^* equipped with the topology $\sigma(E'^*, E')$.*

Let $\mathscr{L}(\bar{E}'; E)$[18] be the space of kernels relative to E. Then the set $\mathscr{L}^+(E) = \mathscr{L}^+(\bar{E}'; \bar{E})$ of non-negative kernels of E is a salient convex cone of $\mathscr{L}(\bar{E}'; E)$ (it is salient because, if H and $-H$ are non-negative, we have $\langle H\bar{e}', e'\rangle = 0$ for all $e' \in E'$, so $\langle H\bar{f}', e'\rangle = 0$ for any e' and f' in E', according to the relation

$$\begin{aligned} 4\langle e', H\bar{f}'\rangle &= \langle e' + f', H\overline{(e' + f')}\rangle - \langle e' - f', H\overline{(e' - f')}\rangle \\ &\quad + i\langle e' + if', H\overline{(e' + if')}\rangle - i\langle e' - if', H\overline{(e' - if')}\rangle, \end{aligned} \tag{3.7}$$

so $H\bar{f}' = 0$ for all f', so $H = 0$).

We are precisely going to show that there is a natural isomorphism between $\mathrm{Hilb}(E)$ and $\mathscr{L}^+(E)$.

Unless explicitly stated otherwise, *we will equip $\mathscr{L}(\bar{E}'; E)$ with the topology of weak pointwise convergence (the coarsest for which each map $H \mapsto \langle H\bar{f}', e'\rangle$ is continuous).*

[18] In this notation, $\bar{E}'$ and E are assumed to be equipped with topologies $\sigma(\bar{E}', \bar{E})$ and $\sigma(E, E')$.

§4. The kernel of a Hilbert subspace of E: the canonical map from $\mathrm{Hilb}(E)$ into $\mathscr{L}^+(E)$

Let $\mathscr{H}$ be a Hilbert subspace of E, and j its inclusion in E. Then the adjoint j^* is a weakly continuous linear map from $\bar{E}'$ into $\bar{\mathscr{H}}'$. Let us call θ the canonical isomorphism from $\bar{\mathscr{H}}'$ onto $\mathscr{H}$. Then $j\theta j^*$:

$$\bar{E}' \xrightarrow{j^*} \bar{\mathscr{H}}' \xrightarrow{\theta} \mathscr{H} \xrightarrow{j} E \tag{4.1}$$

is a weakly continuous linear map H from $\bar{E}'$ into E; we will say that it is the kernel of $\mathscr{H}$, or the kernel associated with $\mathscr{H}$. We will note that in fact, *it maps $\bar{E}'$ into $\mathscr{H}$, and that it is weakly and strongly continuous from $\bar{E}'$ into $\mathscr{H}$*; we will often (if it does not introduce causes for error) interchange $j\theta j^*$ and θj^*. If, moreover, we identify $\bar{\mathscr{H}}'$ and $\mathscr{H}$, then H, as an operator $\bar{E}' \to \mathscr{H}$, is nothing but j^*.

Remark. If E is Hilbertisable, that is to say, if it admits Hilbert structures that define its topology, and if $\mathscr{H}$ is E equipped with one of these Hilbert structures, H is the canonical isomorphism from $\bar{E}'$ onto E defined by $\mathscr{H}$. Conversely, if H is the kernel associated with a Hilbert subspace $\mathscr{H}$ of E, and if $H(\bar{E}') = E$, we necessarily have $\mathscr{H} = E$, so E is Hilbertisable, and $\mathscr{H}$ is a Hilbert structure on E.

Proposition 6. *The kernel H of $\mathscr{H}$ is the unique map from $\bar{E}'$ into $\mathscr{H}$ such that*

$$\forall e' \in E', \quad \forall h \in \mathscr{H}, \quad (h \mid H\bar{e}')_{\mathscr{H}} = \langle h, e' \rangle (= (h \mid \bar{e}')). \tag{4.2}$$

In particular, for $e' \in E'$, $f' \in E'$, we have:

$$(H\bar{f}' \mid H\bar{e}')_{\mathscr{H}} = \langle H\bar{f}', e' \rangle, \quad \textit{so} \tag{4.3}$$

$$\left\|H\bar{e}'\right\|^2_{\mathscr{H}} = \langle H\bar{e}', e' \rangle \geqq 0 : \tag{4.4}$$

whence H is a non-negative kernel.

Proof. We have

$$\begin{aligned} \langle h, e' \rangle &= \langle jh, e' \rangle_{E,E'} = (jh \mid \bar{e}')_{E,\bar{E}'} \\ &= (h \mid j^*\bar{e}')_{\mathscr{H},\bar{\mathscr{H}}'} = (h \mid \theta j^*\bar{e}')_{\mathscr{H}} = (h \mid H\bar{e}')_{\mathscr{H}} \end{aligned} \tag{4.5}$$

that is to say, (4.2); $H\bar{e}'$ is, of course, for a given $\bar{e}'$, the only element of $\mathscr{H}$ to satisfy this equality for any $h \in \mathscr{H}$ since this equality determines its scalar product with every element h of $\mathscr{H}$. We obtain (4.3) by taking $h = H\bar{f}'$ (but (4.3) *is no longer a characterisation of H, because, for example, $H = 0$ satisfies (4.3) for any $e' \in E'$ and $f' \in E'$*).

By setting $f' = e'$, we obtain (4.4), which shows that $H \geqq 0$, as required. □

So $\mathscr{H} \to H$ is a map from $\mathrm{Hilb}(E)$ into $\mathscr{L}^+(E)$ which we will call the canonical map; we will show that this is an isomorphism.

Proposition 7. *The image $H(\bar{E}')$ is a dense subspace $\mathscr{H}_0$ of $\mathscr{H}$. We can characterise it as follows: $h \in \mathscr{H}$ belongs to $\mathscr{H}_0$ if and only if the linear form $k \mapsto (k \mid h)_{\mathscr{H}}$ is continuous on $\mathscr{H}$ with respect to the topology induced by the initial topology or the weak topology of E.*

Proof. As j is injective, $j^*\bar{E}'$ is weakly dense in $\bar{\mathscr{H}}'$[19], so strongly dense since $\mathscr{H}$ is reflexive; as θ is a homeomorphism, $\theta j^*(\bar{E}') = H(\bar{E}')$ is dense in $\mathscr{H}$ (we can also say: if $h \in \mathscr{H}$ is orthogonal to $H(\bar{E}')$, (4.2) shows that $\langle h, e' \rangle = 0$ for all e', so $h = 0$).

If h is of the form $H\bar{e}'$, (4.2) (where we replace h by k) shows that the linear form $k \mapsto (k \mid h) = \langle k, e' \rangle$ is continuous on $\mathscr{H}$ with respect to the topology induced by the weak topology of E. Conversely, if it is continuous with respect to the topology induced by E, the Hahn-Banach Theorem shows that there exists an element e' of E' such that

$$\forall k \in \mathscr{H}, (k \mid h)_{\mathscr{H}} = \langle k, e' \rangle; \tag{4.5b}$$

but $\langle k, e' \rangle = (k \mid H\bar{e}')_{\mathscr{H}}$, so we must have $h = H\bar{e}'$, which proves the Proposition. □

Corollary. *If E' is weakly separable (that is to say, if it admits a weakly dense countable subset), every Hilbert subspace of E is separable.*

Proof. Let D be a countable weakly total subset of E. Since H is weakly continuous and surjective from $\bar{E}'$ onto $\mathscr{H}_0$, $H(\bar{D})$ is weakly total, so strongly total, in $\mathscr{H}_0$; as $\mathscr{H}_0$ is dense in $\mathscr{H}$, $H(\bar{D})$ is total in $\mathscr{H}$, and $\mathscr{H}$ is separable. □

Remark. If E is a separable Banach space, its dual E' is weakly separable. More generally, if E is a separable space, and if $\{0\}$ is a countable intersection of neighbourhoods, then E' is weakly separable. Indeed, let A' be a weakly closed equicontinuous subset of E'. On A', the uniform structure of weak convergence, that is, of pointwise convergence on E, is identical to that of convergence on a dense subset of E, by Ascoli's Theorem[20]; then, since E has a countable dense subset, A' has a uniform structure with a countable entourage basis, and hence is metrisable. Moreover, A' is weakly compact, also by Ascoli; and a compact metrisable space is separable. Thus, every weakly closed equicontinuous subset is weakly separable. But, since $\{0\}$ is the intersection of a sequence V_n of neighbourhoods of 0 in E, we have that E' is the weak closure of the balanced convex envelope of the union of the V_n^0[21a]; each V_n^0 is weakly closed and equicontinuous, so weakly separable by what we just saw, which means that the balanced convex envelope of their union, and hence E', are also weakly separable[21b].

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(19) Bourbaki [1], Chapter IV, §4, n°1, Proposition 3. As $\mathscr{H}$ is reflexive, the dual of $\bar{\mathscr{H}}'$ is $\bar{\mathscr{H}}$; a vector subspace of $\bar{\mathscr{H}}'$ which is dense with respect to $\sigma(\bar{\mathscr{H}}', \bar{\mathscr{H}})$ is then strongly dense in $\bar{\mathscr{H}}'$ (Bourbaki [1], Chapter IV, §2, n°3, Corollary 1 of Proposition 4).

(20) For Ascoli's theorems, see Bourbaki [1], Chapter III, §3, n°5, Proposition 5. A uniform space with a countable base of entourages is metrisable, by Weil's Theorem; Bourbaki [2], §2, n°4, Theorem 1.

(21a) Bourbaki [1], Chapter IV, §1, n°3, Corollary of Proposition 3.

(21b) All this argument was done in Bourbaki [1], Chapter IV, §2, n°2, Corollary of Proposition 3; but this Corollary introduces a hypothesis that is too strong (that E is metrisable).

For example, the spaces $E = \mathscr{D}, \mathscr{E}, \mathscr{S}$ from distribution theory are separable, in which $\{0\}$ is a countable intersection of neighbourhoods[(22)]; so E' is weakly separable, and even strongly separable because these E are reflexive. Actually, $\mathscr{D}$ is enough for us; then, every Hilbert subspace of a space of distributions is a Hilbert subspace of $\mathscr{D}'$, and so is separable.

It is also worth noting that the previous Corollary has nothing to do with Hilbert spaces. For example, *if* $\mathscr{H}$ is a Banach subspace of E with continuous inclusion, if $\mathscr{H}$ is *reflexive, and if* E' *is weakly separable, then* $\mathscr{H}$ *is separable.* Indeed, from the injectivity of $\mathscr{H} \to E$, we can deduce that the transpose $E' \to \mathscr{H}'$ has a weakly dense image; since E' is weakly separable, $\mathscr{H}'$ is weakly separable, so strongly separable since $\mathscr{H}$ is reflexive, so its dual $\mathscr{H}$ is weakly separable as seen above, so separable with respect to its initial topology. The conclusion does not hold if $\mathscr{H}$ is not reflexive; for example, L^∞, a non-reflexive Banach subspace of $\mathscr{D}'$, is not separable.

Proposition 7b. *For $\mathscr{H}_0 = H(\bar{E}')$ to coincide with $\mathscr{H}$, it is necessary and sufficient that the weak topology of $\mathscr{H}$ is induced by the weak topology of E. If E is a Fréchet space, it is necessary and sufficient that the initial topology of $\mathscr{H}$ is induced by that of E; and it is also necessary and sufficient that $\mathscr{H}$ or $\mathscr{H}_0$ is closed in E, or that H is a weak homomorphism (or a strong homomorphism, if E is reflexive)*[(23)].

Proof. Going back to the definition (4.1) of H, $\mathscr{H}_0 = \mathscr{H}$ is equivalent to saying that $j^*(\bar{E}') = \bar{\mathscr{H}}'$. As $j^*(\bar{E}')$ is dense because j is an injection[(24)], this is equivalent to saying that $j^*(\bar{E}')$ is closed in $\bar{\mathscr{H}}'$, and hence that j is a weak homomorphism[(25)], which in turn is equivalent to saying that the weak topology of $\mathscr{H}$ is induced by the weak topology of E.

Let us suppose that E is a Fréchet space. Then j is a weak homomorphism if and only if it is a strong homomorphism, that is, if the initial topology of $\mathscr{H}$ is induced by the initial topology of E; or if and only if $j(\mathscr{H}) = \mathscr{H}$ is closed in E[(26)]. If j is such a homomorphism, it is a monomorphism; as θ is an isomorphism, and j^* is a weak epimorphism (strong if E is reflexive) as the adjoint of a monomorphism[(27)], $H = j\theta j^*$ is a weak homomorphism from $\bar{E}'$ into E (a strong homomorphism if E is reflexive); moreover, $\mathscr{H}_0 = \mathscr{H}$ is closed in E. Conversely, if H is a weak homomorphism, θj^* is a fortiori a weak homomorphism, and as θ is an isomorphism, j^* is a weak homomorphism, so j

(22) The polynomials are dense in $\mathscr{E}$, so $\mathscr{E}$ is separable. Each of its subspaces $\mathscr{D}_K$, with K a compact subset of $\mathbb{R}^n$, is thus separable; so $\mathscr{D}$, as a countable union of $\mathscr{D}_K$, is separable. Finally, $\mathscr{D}$ is dense in $\mathscr{S}$ which is thus separable. In each of these spaces, $\{0\}$ is the intersection of the sequence of neighbourhoods $V_m = \{\phi; |D^p\phi(x)| \leqq 1/m \text{ for } |x| \leqq m, |p| \leqq m\}$.

(23) We call u a homomorphism from E into F if it is continuous and linear, and if the image of an open subset of E is an open subset of $u(E)$. A monomorphism is an injective homomorphism, and an epimorphism is a surjective homomorphism.

(24) See footnote (19), page 20.

(25) Bourbaki [1], Chapter IV, §4, n°1, Proposition 4.

(26) Dieudonné-Schwartz [1], Theorem 7, page 92. Bourbaki [1], Chapter I, §3, n°3, Corollary 3 of Theorem 1.

(27) The transpose (or adjoint) of a homomorphism is a weak homomorphism of a weakly closed image (Dieudonné-Schwartz [1], Theorem 7, page 92); the transpose of an injective map has a weakly dense image (footnote (19) on page 20) so the transpose of a monomorphism is a weak epimorphism. But j^* sends $\bar{E}'$ into $\bar{\mathscr{H}}'$, metrisable of the dual $\bar{\mathscr{H}}$; so if it is a weak homomorphism it is a strong homomorphism (Bourbaki [1], Chapter IV, §4, n°2, Exercise 3b).

is also a weak homomorphism[28]. If furthermore $\mathscr{H}_0$ is closed in E, it is closed in $\mathscr{H}$, and as it is dense, $\mathscr{H}_0 = \mathscr{H}$. □

Proposition 8. *Let $\mathscr{H}$ be a Hilbert subspace of E, and H its kernel. Then an element h of E'^* (the algebraic dual of E' or the weak completion of E) belongs to $\mathscr{H}$ if and only if*

$$\sup_{e' \in E'} \frac{|\langle h, e' \rangle|}{\langle H\bar{e}', e' \rangle^{1/2}} < \infty^{(29)}; \tag{4.6}$$

moreover, in this case, we have:

$$\sup_{e' \in E'} \frac{|\langle h, e' \rangle|}{\langle H\bar{e}', e' \rangle^{1/2}} = \|h\|_{\mathscr{H}}. \tag{4.7}$$

Proof. 1°) If $h \in \mathscr{H}$, we have, according to (4.2) and (4.4):

$$|\langle h, e' \rangle| = |(h \mid H\bar{e}')_{\mathscr{H}}| \leqq \|h\|_{\mathscr{H}} \|H\bar{e}'\|_{\mathscr{H}} = \|h\|_{\mathscr{H}} \langle H\bar{e}', e' \rangle^{1/2}; \tag{4.8}$$

then the left-hand side of (4.6), which we denote by $l(h)$, is bounded above by $\|h\|_{\mathscr{H}}$.

2°) Let us assume conversely that $l(h) < \infty$. The antilinear form $\bar{e}' \mapsto \langle h, e' \rangle$ is then zero on the set of $\bar{e}'$ such that $H\bar{e}' = 0$; this then defines an antilinear form $H\bar{e}' \mapsto \langle h, e' \rangle$ on $H(\bar{E}') = \mathscr{H}_0$. Moreover, its norm is $\leqq l(h)$. So there exists an element k of $\mathscr{H}$ such that

$$\langle h, e' \rangle = (k \mid H\bar{e}')_{\mathscr{H}}; \quad \text{and } \|k\| \leqq l(h). \tag{4.9}$$

The equality (4.2) applied to k then shows that

$$\langle h, e' \rangle = \langle k, e' \rangle \text{ for all } e' \in E', \tag{4.10}$$

so $k = h$, which proves that $h \in \mathscr{H}$ and that $\|h\|_{\mathscr{H}} \leqq l(h)$, or $= l(h)$ by 1°).

□

Remark. It is convenient to set $\|h\|_{\mathscr{H}} = \infty$ for $h \notin \mathscr{H}$; then (4.7) holds for all h in E'^*.

Corollary 1. *The canonical map from* Hilb(E) *into* $\mathscr{L}^+(E)$ *is injective.*

Indeed, Proposition 8 shows that the knowledge of H determines $\mathscr{H}$, with its Hilbert structure.

Corollary 2. *If $h \in E'^*$ satisfies (4.6), then $h \in E$.*

Proposition 9. *Let B_0 be the set of $H\bar{e}'$, $e' \in E'$, such that $\langle H\bar{e}', e' \rangle \leqq 1$, and $\bar{B}_0$ its closure in E.*

Then

$$\mathscr{H} = \bigcup_{\lambda \in \mathbb{R}^+} \lambda \bar{B}_0.$$

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[28] Dieudonné-Schwartz [1], Theorem 7, page 92.

[29] If the denominator is zero and the numerator is not for an element e' the supremum is of course infinite; if both are zero, then we do not take this element e' into account for the calculation of the supremum.

Proof. B_0 is the unit ball of $\mathscr{H}_0$; as $\mathscr{H}_0$ is dense in $\mathscr{H}$ (Proposition 7), its closure B in $\mathscr{H}$ is the unit ball of $\mathscr{H}$, and, as this is weakly compact in $\mathscr{H}$ and hence in E, it is closed in E, and thus coincides with $\bar{B}_0$, the closure of B_0 in E. As then we have $\mathscr{H} = \bigcup_{\lambda\in\mathbb{R}^+} \lambda B$, we have $\mathscr{H} = \bigcup_{\lambda\in\mathbb{R}^+} \lambda \bar{B}_0$. □

Remark. This gives a new characterisation of $\mathscr{H}$ starting from H, thereby reproving the Corollary of Proposition 8.

Example 1. Let us consider Example 3 on page 8. For $\bar{\mathscr{D}}$, let us choose $\mathscr{D}$ itself, and for conjugation the usual conjugation of forms, $\phi \mapsto \bar{\phi}$ (which commutes with *). Then by setting $h = \alpha$ and $e' = {}^*\beta$ in (4.2), (1.2) shows that $H\overline{{}^*\beta} = \beta$, so that $H\phi = {}^{*-1}\bar{\phi}$; H is indeed a continuous linear map from $\bar{E}' = \mathscr{D}$ into $E = \mathscr{D}'$.

We can also take $\bar{\mathscr{D}} = \mathscr{D}$, the conjugation this time being ${}^{*-1}$: $\phi \mapsto {}^{*-1}\phi$. *But this is not an anti-involution*: for a form ϕ of degree $n-p$, ${}^{*-1*-1}\phi = (-1)^{p(n-p)}\phi$. We could customise the degrees, to avoid errors.

We will consider $\overset{p}{\mathscr{H}} \subset \overset{p}{\mathscr{D}}{}' = E$; for E', we will take the space $\overset{n-p}{\mathscr{D}}$, with $\langle \overset{p}{T}, \overset{n-p}{\phi} \rangle = \int_X T \wedge \phi$. For $\bar{E}'$, we will then take the space $\overset{p}{\mathscr{D}}$, with the anti-isomorphism $\phi \mapsto {}^{*-1}\phi$, from $\overset{n-p}{\mathscr{D}}$ onto $\overset{p}{\mathscr{D}}$; there is no question of an involution here, since $E' \neq \bar{E}'$. The product $(T \mid \phi)$ following (0.4) ($T = e$, $\phi = \bar{e}' = {}^{*-1}\psi$, with $e' = \psi = {}^*\phi$) is then $\langle T, {}^*\phi \rangle$; it is this that one considers in the theory of Hodge-de Rham.

Then, by applying (4.2) with $h = \alpha$ and $e' = {}^*\beta$ again, (1.2) shows that $H({}^{*-1*}\beta) = \beta$ or $H\beta = \beta$; H is the identity, or more precisely the inclusion of $\overset{p}{\mathscr{D}}$ in $\overset{p}{\mathscr{D}}{}'$; it is the inclusion of $\mathscr{D}$ in $\mathscr{D}'$ if the degrees are not specified.

In this example, H is not the same operator from $\mathscr{D}$ into $\mathscr{D}'$, depending on whether we choose the usual conjugation or ${}^{*-1}$; this is not surprising, since H is an operator from $\bar{\mathscr{D}}$ into $\mathscr{D}'$, and not from $\mathscr{D}$ into $\mathscr{D}'$.

Example 1b. Let us assume that X is an open subset of $\mathbb{R}^n$ and let us take $\mathscr{H} = L^2(X)$. Let us take the usual conjugation. We have

$$(f \mid \phi)_{L^2} = \int_X f(x)\bar{\phi}(x)dx = \langle f, \bar{\phi} \rangle;$$

this is (4.2) with $H\bar{\phi} = \bar{\phi}$, and H is the identity map, or the natural inclusion of $\bar{E}' = \mathscr{D}$ in $E = \mathscr{D}'$ (a particular case of what we just saw in Example 1a for degree $p = 0$, if we identify forms of degree 0 and n by the Lebesgue measure).

Example 1c. Let us return to Example 4 on page 9. Equation (1.2c), where we replace ϕ by $\bar{\phi}$, is Equation (4.2) with $h = f\mu$, $e' = \bar{\phi}$ and $\bar{e}' = \phi$, if we take $H\phi = \phi\mu$, which indeed defines H as a continuous linear map from $\bar{E}' = \mathscr{D}^\circ(X)$ (or $\mathscr{D}(X)$ if X is an open subset of $\mathbb{R}^n$) into $E = \mathscr{D}'^0_c(X)$ (or $\mathscr{D}'(X)$).

Example 2. Let us return to Example 2 on page 8. Let us denote by D the differential operator

$$D = \sum_p (-1)^{|p|} a_p D^{2p}. \tag{4.11}$$

Then we show that the kernels of Hilbert subspaces $\mathscr{H}^{-s}(X)$, $\mathscr{H}_0^s(X)$ and $\mathscr{H}^s(X)$ of $\mathscr{D}'(X)$ (where $s \geqq 0$ is a natural number) are respectively the differential operator D, Green's operator and de Neumann's operator on an open subset X of $\mathbb{R}^n$, all considered as operators from $\bar{E}' = \mathscr{D}(X)$ into $E = \mathscr{D}'(X)$. We could show it now, but we prefer to do it later with appropriate methods (Examples on pages 70 and 73).

Example 3. Let $E = \mathscr{D}'(X)$, the space of distributions on an open subset X of $\mathbb{R}^n$. Then $E' = \mathscr{D}(X)$; the complex conjugation establishes contragredient anti-involutions on these spaces, each of which corresponds to its own conjugate space. A kernel H is then a continuous linear map from $\mathscr{D}$ into $\mathscr{D}'$; the Kernel Theorem(30) says that it is a distribution $H_{x,\xi}$ on $X \times X$, the map H from $\mathscr{D}$ into $\mathscr{D}'$ defined by

$$\phi \mapsto (H \cdot \phi)_x = \int_X H_{x,\xi} \phi(\xi) d\xi. \tag{4.12}$$

The conjugate $\bar{H}$ is defined by the conjugate distribution $\bar{H}_{x,\xi}$, the transpose ${}^tH$ by the symmetric distribution $({}^tH)_{x,\xi} = H_{\xi,x}$, and the adjoint by $(H^*)_{x,\xi} = \overline{H_{\xi,x}}$; a kernel H is Hermitian if $H_{x,\xi} = \overline{H_{\xi,x}}$. A kernel H is of positive type if

$$\forall \phi \in \mathscr{D}(X), \qquad \int\int_{X \times X} H_{x,\xi} \phi(x) \overline{\phi(\xi)} dx d\xi \geqq 0; \tag{4.13}$$

then it is Hermitian, and $\bar{H}$ is also of positive type.

If $\mathscr{H}$ is a Hilbert subspace of $\mathscr{D}'(X)$, its kernel H is defined by (4.2):

$$\begin{aligned} &\forall \phi \in \mathscr{D}, \forall T \in \mathscr{H}, \qquad (T \mid H \cdot \bar{\phi})_{\mathscr{H}} = \langle T, \phi \rangle \qquad \text{or} \\ &(T \mid H \cdot \phi)_{\mathscr{H}} = \langle T, \bar{\phi} \rangle = (T \mid \phi). \end{aligned} \tag{4.14}$$

Here, (4.3) and (4.4) become:

$$\begin{aligned} \forall \phi \in \mathscr{D}, \forall \psi \in \mathscr{D}, (H \cdot \bar{\psi} \mid H \cdot \bar{\phi})_{\mathcal{H}} &= \langle H \cdot \bar{\psi}, \phi \rangle \\ &= \int\int_{X \times X} H_{x,\xi} \phi(x) \overline{\psi(\xi)} dx d\xi. \end{aligned} \tag{4.15}$$

$$\forall \phi \in \mathscr{D}, \left\| H \cdot \bar{\phi} \right\|_{\mathscr{H}}^2 = \int\int_{X \times X} H_{x,\xi} \phi(x) \bar{\phi}(\xi) dx d\xi \geqq 0. \tag{4.16}$$

We will also use them in the form

$$\begin{aligned} (H \cdot \phi \mid H \cdot \psi)_{\mathscr{H}} &= \langle H \cdot \phi, \bar{\psi} \rangle \\ \|H \cdot \phi\|_{\mathscr{H}}^2 = \langle H \cdot \phi, \bar{\phi} \rangle &= \int\int_X H_{x,\xi} \overline{\phi(x)} \phi(\xi) dx d\xi. \end{aligned} \tag{4.17}$$

Now let $X = \mathbb{R}^n$, and E one of the following subspaces of $\mathscr{D}'(\mathbb{R}^n)$: $\mathscr{D}'$, $\mathscr{D}$, $\mathscr{E}'$, $\mathscr{E}$, $\mathscr{S}'$, $\mathscr{S}$, $\mathscr{O}'_M$, $\mathscr{O}_M$, $\mathscr{O}'_C$, $\mathscr{O}_C$(31). Every weakly continuous linear map from E' into E is strongly continuous(32), hence continuous from E'_c into E (in all

(30) Kernel Theorem, Schwartz [2], page 143 and Schwartz [3], Chapter I, §4, Proposition 25.
(31) For all these spaces of distributions, see Schwartz [1].
(32) See footnote (17) on page 17.

these spaces E, the bounded subsets of E are relatively compact, so $E'_c = E'$ with the strong topology), and conversely, every continuous map from E'_c into E is weakly continuous from E' into E[(33)]. $\mathscr{L}(E'; E) = E\varepsilon E = E\hat{\otimes}_\varepsilon E$, because these spaces are complete and have the approximation property[(34)]. Moreover, for all these spaces *except* $\mathscr{D}$, $E_x \hat{\otimes}_\varepsilon E_\xi$ is the space $E_{x,\xi}$ analogous to $\mathbb{R}^n \times \mathbb{R}^n$[(35)], which gives the structure of the corresponding kernels; for a Hilbert subspace $\mathscr{H}$ of E, we will have Equations (4.12) to (4.17), for $\phi, \psi ... \in E$. For example, if $\mathscr{H}$ is a Hilbert subspace of $\mathscr{S}'(\mathbb{R}^n)$, H will be tempered, $H \in \mathscr{S}'_{x,\xi}$, and the preceding formulae will be correct for $\phi, \psi ...$ in $\mathscr{S}$.

Example 4. Let E be a Hilbert space. We can identify $\bar{E}'$ with E (see Example 4 on page 4). Then a kernel H is simply a continuous linear operator from E into itself. Its adjoint H^* is what we usually call the adjoint, and the concepts of Hermitian and positive operators are the usual concepts thereof.

Let $\mathscr{H}$ be a Hilbert subspace of E (beware, there is a possible confusion here. This is a Hilbert subspace in the sense of §1, but it does not necessarily have the structure induced by that of E!). Its kernel H is a non-negative Hermitian continuous linear operator from E into E, which is defined as follows. Let j be the inclusion $\mathscr{H} \to E$ and j^* its adjoint $E \to \mathscr{H}$ (identifying $\bar{\mathscr{H}}'$ with $\mathscr{H}$). Equation (4.1) becomes

$$E \xrightarrow{j^*} \mathscr{H} \xrightarrow{j} E, H = jj^*, \tag{4.18}$$

which, by abuse of notation, we also often write as j^* (see page 19), a map from E into $\mathscr{H}$. H is characterised by (4.2):

$$\forall h \in \mathscr{H}, \forall \xi \in E, (h \mid H\xi)_{\mathscr{H}} = (h \mid \xi)_E. \tag{4.19}$$

(4.3) and (4.4) become:

$$\begin{aligned} (H\xi \mid H\eta)_{\mathscr{H}} &= (H\xi \mid \eta)_E \\ \|H\xi\|^2_{\mathscr{H}} &= (H\xi \mid \xi)_E \geqq 0. \end{aligned} \tag{4.20}$$

The subspace $\mathscr{H}$ is closed in E if and only if H is a homomorphism (Proposition 1b), and then $\mathscr{H} = \mathscr{H}_0 = H(E)$. We have $\mathscr{H} = E$ (*as sets*) if and only if H is invertible (because, as an application from $\bar{E}'$ into E, H has to be the canonical isomorphism from $\bar{E}'$ onto E with respect to the Hilbert structure $\mathscr{H}$; see Remark on page 19). If $\mathscr{H}$ is a closed subspace of E with the induced Hilbert structure, H is the orthogonal projection onto $\mathscr{H}$, as shown in (4.19).

Proposition 9b. *Let $\mathscr{K}$ be a Hilbert subspace of E, with kernel K, considered as an operator from $\bar{E}'$ into $\mathscr{K}$; let $\mathscr{H}$ be a Hilbert subspace of $\mathscr{K}$, and let A be its kernel of $\mathscr{H}$ in the sense of Example 4 above, an operator from $\mathscr{K}$ into $\mathscr{H}$; then the kernel H of $\mathscr{H}$ with respect to E, as an operator from $\bar{E}'$ into $\mathscr{H}$, is $H = AK$: $\bar{E}' \xrightarrow{K} \mathscr{K} \xrightarrow{A} \mathscr{H}$.*

(33) A continuous linear map from E'_c into E is continuous with respect to $\sigma(E'_c, (E'_c)')$ and $\sigma(E, E')$ by Bourbaki [1], Chapter IV, §4, n°2, Proposition 6; but $(E'_c)' = E$, Bourbaki [1], Chapter IV, §2, n°3, Corollary of Theorem 2.

(34) For the product ε, the tensor product $\hat{\otimes}_\varepsilon$ and the approximation property, see Schwartz [3], Preliminaries and Chapter I, §1.

(35) Schwartz [3], Chapter I, §4, Proposition 28, page 98.

Proof. Let us also identify $\bar{\mathscr{H}}'$ with $\mathscr{H}$. Then K and H are maps $\bar{E}' \to \mathscr{K}$ and $\bar{E}' \to \mathscr{H}$, adjoints of the inclusions $\mathscr{K} \to E$ and $\mathscr{H} \to E$. But $\mathscr{H} \to E$ factorises into $\mathscr{H} \to \mathscr{K} \to E$, so $\bar{E}' \xrightarrow{H} \mathscr{H}$ factorises into $\bar{E}' \xrightarrow{K} \mathscr{K} \to \mathscr{H}$, and $\mathscr{K} \to \mathscr{H}$ is precisely A. □

Corollary. *If $\mathscr{H}$ is a closed subspace of $\mathscr{K}$, with the induced Hilbert structure, H is equal to AK, where A is the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$.*

It suffices to apply Proposition 9b to Example 4.

Proposition 9c. *Let $\mathscr{H}$ be a Hilbert subspace of E, with kernel H. For the inclusion j of H in E to be compact (in other words, for the unit ball of $\mathscr{H}$ to be compact in E), it is necessary for H to belong to $\mathscr{H}\hat{\otimes}_\varepsilon \bar{E}$, and it is sufficient for it to belong to $E\varepsilon\bar{E}$*[36].

Proof. Saying that j is compact is to say that the unit ball B of $\mathscr{H}$ is *relatively compact* in E; but it is weakly compact in $\mathscr{H}$ and hence in E, and so it is *compact* in E.

1°) Let us assume that j is compact. Then, by polarity, the image under ${}^tj$ of the polar set of B in E' is contained in the polar set of B in $\mathscr{H}'$, so ${}^tj$ is continuous from E'_c into $\mathscr{H}'$, and j^* is continuous from $\bar{E}'_c$ into $\bar{\mathscr{H}}'$; then $H = \theta j^*$, as an operator from $\bar{E}'$ into $\mathscr{H}$, is continuous from $\bar{E}'$ into $\mathscr{H}$, so $H \in \mathscr{H}\varepsilon\bar{E}$; as the Hilbert space $\mathscr{H}$ has the strict approximation property, $H \in \mathscr{H}\hat{\otimes}_\varepsilon\bar{E}$, as claimed in the statement. Naturally $H = j\theta j^*$ is the image under $j \otimes I$ of $\theta j^* \in \mathscr{H}\hat{\otimes}_\varepsilon E$, so belongs to $E\hat{\otimes}_\varepsilon\bar{E} \subset E\varepsilon\bar{E}$.

2°) Conversely, let us assume that $H \in E\varepsilon\bar{E} = \mathscr{L}(\bar{E}'_c; E)$. Let A' be a compact weakly equicontinuous subset of E', and $\bar{A}'$ its conjugate in $\bar{E}'$. As H is in $E\varepsilon\bar{E}$, $H(\bar{A}')$ is compact in E[37], and the restriction of H to $\bar{A}'$ is continuous from $\bar{E}'$ with the weak topology into E with the strong topology. Let us assume that $\bar{e}' \in \bar{A}'$ converges weakly to e'_0; then

$$\begin{aligned}(4.21)\qquad \langle H\bar{e}', e'\rangle &- \langle H\overline{e'_0}, e'_0\rangle \\ &= \langle H\overline{(e'-e'_0)}, e'\rangle + \langle H\bar{e}'_0, e'-e'_0\rangle;\end{aligned}$$

$H\overline{(e'-e'_0)}$ converges strongly to 0, and $e' \in A'$ is equicontinuous, so the first term converges to 0; the second term also converges to 0, since e' converges weakly to e'_0; so $\langle H\bar{e}', e'\rangle$ converges to $\langle H\bar{e}'_0, e'_0\rangle$. This proves that $H\bar{e}'$, which converges weakly to $H\bar{e}'_0$ in $\mathscr{H}$, has norm $\langle H\bar{e}', e'\rangle^{1/2}$ which converges to that of $H\bar{e}'_0$, so $H\bar{e}'$ converges to $H\bar{e}'_0$ *strongly* in $\mathscr{H}$[38]. Thus $H = \theta j^*$ is continuous from $\bar{A}'$ (equipped with the weak

[36] See Schwartz [3], Chapter I, Proposition 11 and Corollary 1. A Hilbert space $\mathscr{H}$ satisfies the strict approximation condition, because every operator u from $\mathscr{H}$ into $\mathscr{H}$ is an adherent point in $\mathscr{L}_c(\mathscr{H};\mathscr{H})$ to $P\cdot u$, where the P are orthogonal projections of finite rank and $\|P\cdot u\| \leqq \|u\|$.

[37] Schwartz [3], Chapter I, §1, Proposition 5, page 35.

[38] In a Hilbert space, weak convergence with convergence in norm implies strong convergence. Indeed, if x_j converges weakly to x, $\frac{1}{2}(x+x_j)$ also converges weakly to x, so $\liminf\|\frac{1}{2}(x+x_j)\| \geqq \|x\|$; but if $\|x_j\|$ converges to $\|x\|$, we also have $\limsup\|\frac{1}{2}(x+x_j)\| \leqq \|x\|$; so $\lim\|\frac{1}{2}(x+x_j)\| = \|x\|$. But then Apollonius's Theorem $\|x_j\|^2 + \|x\|^2 = 2(\|\frac{1}{2}(x+x_j)\|^2 + \frac{1}{2}\|x-x_j\|^2)$ shows that x_j converges strongly to x.

topology) into $\mathscr{H}$ with the strong topology; so $H(\bar{A}')$ is compact in $\mathscr{H}$. As H is weakly continuous from $\bar{E}'$ into $\mathscr{H}$, this proves that H belongs to $\mathscr{H}\varepsilon\bar{E} = \mathscr{L}(\bar{E}'_c; \mathscr{H})$. But $H = \theta j^*$, where θ is an isomorphism, so j^* is continuous from $\bar{E}'_c$ into $\bar{\mathscr{H}}'$, and ${}^t j$ continuous from E'_c into $\mathscr{H}'$; by polarity, this proves that $j(B)$ is relatively compact in E, so j is compact.

□

Remark 1. This Proposition proves (along with Proposition 10, which we will show later) that, even if E does not satisfy the strict approximation property, every non-negative element of $E\varepsilon\bar{E}$ (in the sense of the positivity of kernels) is in $E\hat{\otimes}_\varepsilon\bar{E}$.

Remark 2. Let us suppose that, in E, every bounded subset is relatively compact. Then every Hilbert subspace of E has a compact inclusion. But we also have that every kernel H maps the equicontinuous subsets of E' to bounded subsets, hence relatively compact subsets, and the space $\mathscr{L}(\bar{E}'; E)$ of kernels is identical (putting aside the topology) to $E\varepsilon\bar{E}$. It is what we already applied at the end of Example 3 on page 24.

§5. The Hilbert subspace associated to a non-negative kernel. Bijectivity of the canonical map from Hilb(E) into $\mathscr{L}^+(E)$.

Proposition 10. *The canonical map from* Hilb(E) *into* $\mathscr{L}^+(E)$ *is a bijection.*

First Proof. We have already seen (Corollary of Proposition 8, or Remark following Proposition 9) that this map is injective. But we will see this for the third time, and show moreover that it is surjective.

Let H be a non-negative kernel. Let us consider $\mathscr{H}_0 = H(\bar{E}')$ (see Proposition 7). If there exists a Hilbert subspace $\mathscr{H}$ with kernel H, the pre-Hilbert structure induced on $\mathscr{H}_0$ is known, since we have (4.3). Conversely, (4.3) defines a Hermitian form on $\mathscr{H}_0$; for $u \in \mathscr{H}_0$ and $v = H\bar{e}' \in \mathscr{H}_0$, we will indeed put

$$(u \mid v)_{\mathscr{H}_0} = \langle u, e' \rangle; \tag{5.1}$$

the result only depends on $v = H\bar{e}'$ and not on e', because, if $u = H\bar{f}'$, we have

$$\langle u, e' \rangle = \langle H\bar{f}', e' \rangle = \overline{\langle H\bar{e}', f' \rangle} = \overline{\langle v, f' \rangle}, \tag{5.2}$$

because H is Hermitian. This Hermitian form is non-negative, because H is non-negative . Hence, it makes $\mathscr{H}_0$ a pre-Hilbert space. Its topology is finer than that induced by E. To see this, we have to show that the unit ball of $\mathscr{H}_0$,

$$B_0 = \{H\bar{e}', e' \in E'; \langle H\bar{e}', e' \rangle \leqq 1\},$$

is bounded in E, or, equivalently, weakly bounded; this is to say that, for every fixed f' in E', the set of numbers $|\langle H\bar{e}', f' \rangle|$ is bounded for $H\bar{e}' \in B_0$; this is the result of the Cauchy-Schwarz inequality:

$$|\langle H\bar{e}', f' \rangle| \leqq \langle H\bar{e}', e' \rangle^{1/2} \langle H\bar{f}' f' \rangle^{1/2}. \tag{5.3}$$

This result shows us in particular that $\mathscr{H}_0$ is Hausdorff, and hence that its Hermitian form (5.1) is positive definite.

Then, if there exists a $\mathscr{H}$ of kernel H, it is necessarily the completion of $\mathscr{H}_0$ in E (Propositions 7 and 1); so it is unique. To show that such a completion exists, we have to show that the map $\hat{\jmath}$ defined in Proposition 1 is injective. Let us consider the equality

$$(k \mid H\bar{e}')_{\hat{\mathscr{H}}_0} = \langle \hat{\jmath}(k), e' \rangle, \tag{5.4}$$

for $k \in \hat{\mathscr{H}}_0$ and $e' \in E'$; for fixed e', this equality holds if $k \in \mathscr{H}_0$, by (5.1); both sides depend continuously on $k \in \hat{\mathscr{H}}_0$, because $\hat{\jmath}$ is continuous from $\hat{\mathscr{H}}_0$ into E, and because it is a continuous linear form on E; so (5.4) holds for every $e' \in E'$ and $k \in \hat{\mathscr{H}}_0$. Then, if $\hat{\jmath}(k) = 0$, k is orthogonal to $\mathscr{H}_0$ in $\hat{\mathscr{H}}_0$, so k is the zero element, which means $\hat{\jmath}$ is injective. (We can also show the existence of the completion of $\mathscr{H}_0$ in E by using the other criterion given in Proposition 1: the unit ball B_0 is closed in $\mathscr{H}_0$ with respect to the topology induced by E. Indeed, $u \in B_0$ is equivalent to $u \in \mathscr{H}_0$ and

$$|\langle u, f' \rangle| \leqq 1 \quad \text{for every } f' \in E' \text{ such that } H\bar{f}' \in B_0; \tag{5.5}$$

for a given f', the set of u such that $|\langle u, f' \rangle| \leqq 1$ is closed in E, so B_0 is the intersection of $\mathscr{H}_0$ and a set of closed subsets of E, which satisfies the desired criterion).

Then let $\mathscr{H} = \hat{\jmath}(\hat{\mathscr{H}}_0)$ be the completion of $\mathscr{H}_0$ in E. For an $h \in \mathscr{H}$, let k be the element of $\hat{\mathscr{H}}_0$ such that $\hat{\jmath}(k) = h$; (5.4) gives (4.2), because $(k \mid H\bar{e}')_{\hat{\mathscr{H}}_0} = (h \mid H\bar{e}')_{\mathscr{H}}$, so the kernel of $\mathscr{H}$ is H. We have thus shown that given a non-negative kernel H, there exists a Hilbert subspace $\mathscr{H}$, and that the one that admits H as its associated kernel is unique, so $\mathrm{Hilb}(E) \to \mathscr{L}^+(E)$ is bijective. □

It will subsequently be useful to record, in concise terms, this method of constructing $\mathscr{H}$ from H:

(5.5b) *On $\mathscr{H}_0 = H(\bar{E}')$ we define the scalar product (5.1); this makes $\mathscr{H}_0$ a pre-Hilbert subspace of E, and $\mathscr{H}$ is the completion of $\mathscr{H}_0$ in E.*

Second Proof. Of course, this second proof that we are going to give is not essentially distinct from the first. Having abundantly proven the injective character of the canonical map, we will restrict ourselves to proving its surjectivity. Let H be a non-negative kernel. It defines a structure of a pre-Hilbert space on E' (not Hausdorff, in general) by the Hermitian form (3.5); we will denote this structure by E'_H. Then E'_H has a dual $(E'_H)'$, which we will call $\mathscr{H}$, which is a Hilbert space and which is also the dual of the complete Hausdorff space $\hat{E}'_H$ associated to E'_H. $\mathscr{H}$ is a subspace of the algebraic dual E'^* of E' (weak completion of E); it is the set of $h \in E'^*$ which satisfy

$$\begin{aligned} \left|\left\langle h, e' \right\rangle\right| &\leqq \text{constant} \times \left\|e'\right\|_{E'_H} \\ &= \text{constant} \times \tilde{H}(e', e') \\ &= \text{constant} \times \langle H\bar{e}', e' \rangle, \end{aligned} \tag{5.5c}$$

in other words, (4.6). Its topology is finer than $\sigma(E'^*, E')$, hence it is a Hilbert subspace of E'^* with the weak topology. In addition, if $\hat{L}$ is the canonical anti-isomorphism of $\hat{E}'_H$ onto its dual $\mathscr{H}$, then its composition $L : E'_H \to \hat{E}'_H \xrightarrow{\hat{L}} \mathscr{H}$

with the canonical map from E'_H into $\hat{E}'_H$ is continuous and antilinear from E'_H into $\mathscr{H}$ and has the following properties:

1°) the image $L(E'_H) = \mathscr{H}_0$ is dense in $\mathscr{H}$, because the image of E'_H in $\hat{E}'_H$ is dense and $\hat{L}(\hat{E}'_H) = \mathscr{H}$;

2°) for any $h \in \mathscr{H}$ and any $e' \in E'$, denoting the image of e' in $\hat{E}'_H$ as $\dot{e}'$,

$$(5.6)\qquad (h \mid Le')_{\mathscr{H}} = (h \mid \hat{L}\dot{e}')_{\mathscr{H}} = \langle h, \dot{e}'\rangle = \langle h, e'\rangle;$$

3°) for any $e' \in E'$ and $f' \in E'$, with images $\dot{e}'$ and $\dot{f}'$ in $\hat{E}'_H$:

$$(5.7a)\qquad (Lf' \mid Le')_{\mathscr{H}} = (Lf' \mid \hat{L}\dot{e}')_{\mathscr{H}} = \langle Lf', \dot{e}'\rangle = \langle Lf', e'\rangle.$$

(5.7b)

$$(Lf' \mid Le')_{\mathscr{H}} = (\hat{L}\dot{f}' \mid \hat{L}\dot{e}')_{\mathscr{H}} = (\dot{e}' \mid \dot{f}')_{\hat{E}'_H} = (e' \mid f')_{E'_H} = \langle H\bar{f}', e'\rangle.$$

The comparison of (5.7a) and (5.7b) shows that $Lf' = H\bar{f}'$. So $\mathscr{H}_0 = L(E')$ is nothing but $\mathscr{H}_0 = H(\bar{E}')$ already considered previously, and, on $\mathscr{H}_0$, the pre-Hilbert structure induced by $\mathscr{H}$ is what is defined by (4.3) in accordance with (5.7a). So $\mathscr{H}$ is the completion of $\mathscr{H}_0$ in E'^* with the weak topology. Proposition 0 then shows that $\mathscr{H}$ is itself a Hilbert subspace of E. Then (5.6) shows that $\mathscr{H}$ has H as its kernel, and this is the desired Hilbert space. □

The above construction of $\mathscr{H}$ from H can be summed up in the following concise terms: *On E', the kernel H defines, by (3.5), a structure of a pre-Hilbert space (non-Hausdorff, in general) E'_H. $\mathscr{H}$ is the dual of E'_H* (a priori a Hilbert subspace of E'^* with the weak topology, it is in fact a Hilbert subspace of E).

We see the difference between the two proofs: the first never leaves E, but shows that $\mathscr{H}_0$ has a completion in E; the second gives a Hilbert space $\mathscr{H}$ in E'^* to begin with, and we had to show that it is in E.

Corollary. *Every separately weakly continuous non-negative sesquilinear form B on $E' \times E'$ is strongly continuous. Every sesquilinear form A on $E' \times E'$ satisfying the upper bound*

$$(5.8)\qquad \left|A(e', f')\right| \leqq \text{const.}(B(e', e'))^{1/2}(B(f', f'))^{1/2}, \quad e' \in E', f' \in E',$$

or even just the upper bound

$$(5.8b)\qquad \left|A(e', e')\right| \leqq \text{const.}B(e', e'), \quad e' \in E',$$

is separately weakly continuous, and is strongly continuous.

Proof. Let us assume that (5.8) holds. B, being non-negative and separately weakly continuous, is defined by a kernel $H \geqq 0$ following (3.5) (Proposition 5): $B(e', f') = \langle H\bar{f}', e'\rangle$, or $B = \bar{H}$. Then the linear form h on E' defined by $e' \mapsto A(e', f')$, for a fixed f', satisfies (4.6) thanks to (5.8); by Proposition 8, h is in $\mathscr{H}$, which is a Hilbert subspace of E with kernel H, and so h is in E: it is weakly continuous on E'. The same reasoning shows that, for a fixed f', $f' \mapsto \overline{A(e', f')}$ is weakly continuous, so A is separately weakly continuous.

If now e' and f' converge strongly to 0 in E', $H\bar{e}'$ and $H\bar{f}'$ converge strongly to 0 in $\mathscr{H}$, so their scalar product in $\mathscr{H}$, which is $\langle H\bar{f}', e'\rangle = B(e', f')$, converges to 0; moreover, $B(e',e')$ and $B(f',f')$ converge to 0 for the same reason, hence so does $A(e',f')$ by (5.8); A and B are strongly continuous on $E' \times E'$.

Let us now suppose that only (5.8b) holds. Then it suffices to use the following lemma. □

Lemma. *Let A and B be two sesquilinear forms on $E' \times E'$, with $B \geqq 0$. The inequality*

$$\forall e' \in E', \forall f' \in E', \quad \left|A(e',f')\right| \leqq (B(e',e'))^{1/2}(B(f',f'))^{1/2} \tag{5.9}$$

is equivalent, if A is Hermitian, to the inequality

$$\forall e' \in E', \quad \left|A(e',e')\right| \leqq B(e',e'); \tag{5.10}$$

for any A, it is implied by the following inequality:

$$\forall e' \in E', \quad \left|A(e',e')\right| \leqq \frac{1}{2}B(e',e'). \tag{5.11}$$

Proof. (5.9) always implies (5.10). Conversely, let us assume that (5.10) holds and A is Hermitian. We have

$$\begin{aligned} 4\mathrm{Re}A(e',f') &= A(e'+f',e'+f') - A(e'-f',e'-f') \\ \left|4\mathrm{Re}A(e',f')\right| &\leqq B(e'+f',e'+f') + B(e'-f',e'-f') \\ &= 2(B(e',e') + B(f',f')) \end{aligned}$$

By replacing f' by $\lambda f'$, $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and such that $A(e',\lambda f')$ is real, we deduce that

$$\left|A(e',f')\right| \leqq \frac{1}{2}(B(e',e') + B(f',f')), \qquad \forall e' \in E', \forall f' \in E'. \tag{5.12}$$

It remains to show that (5.12) implies (5.9). It is true if $B(e',e')$ or $B(f',f')$ is zero; if indeed, for example, $B(e',e') = 0$, we replace, in (5.12), f' by tf', $t \in \mathbb{C}$, and we obtain $|\bar{t}A(e',f')| \leqq \frac{1}{2}|t|^2 B(f',f')$, which, by letting $t \neq 0$ tend towards 0, gives $A(e',f') = 0$ $\forall f' \in E'$, that is to say, (5.9). If now $B(e',e')$ and $B(f',f')$ are both $\neq 0$, we replace, in (5.12), e' and f' by $\frac{e'}{(B(e',e'))^{1/2}}$ and $\frac{f'}{(B(f',f'))^{1/2}}$ respectively, which gives

$$\frac{\left|A(e',f')\right|}{(B(e',e'))^{1/2}(B(f',f'))^{1/2}} \leqq \frac{1}{2}(1+1) = 1, \tag{5.13}$$

i.e. (5.9).

If now A just satisfies (5.10), we have

$$\begin{aligned} 4A(e',f') =& A(e'+f',e'+f') - A(e'-f',e'-f') \\ &+ iA(e'+if',e'+if') - iA(e'-if',e'-if'); \\ |4A(e',f')| \leq& \frac{1}{2}(B(e'+f',e'+f') + B(e'-f',e'-f') \\ &+ B(e'+if',e'+if') + B(e'-if',e'-if')) \\ =& 2(B(e',e') + B(f',f')), \end{aligned}$$

that is to say, (5.12), from which we obtain (5.9) again. □

Remark. If A does not satisfy an upper bound of type (5.8), it is not in general separately weakly continuous, and, if it is, it is not in general strongly continuous on $E' \times E'$.

§6. Canonical isomorphism of Hilb(E) and $\mathcal{L}^+(E)$

Theorem. *The canonical map* $\mathrm{Hilb}(E) \to \mathscr{L}^+(E)$ *defined in §5 is an isomorphism from* $\mathrm{Hilb}(E)$ *onto* $\mathscr{L}^+(E)$ *with respect to the structures defined on these sets.*

This theorem is the result of 3 propositions.

Proposition 11. *If the Hilbert subspace* $\mathscr{H}$ *of* E *has* H *as its kernel, the subspace* $\lambda\mathscr{H}$*, where* $\lambda \geqq 0$*, has* λH *as its kernel.*

Proof. According to the definition of $\lambda\mathscr{H}$ (Equation (2.1)), we have, for any $h \in \lambda\mathscr{H}$ and $e' \in E'$:

$$(h \mid \lambda H\bar{e}')_{\lambda\mathscr{H}} = \frac{1}{\lambda}(h \mid \lambda H\bar{e}')_{\mathscr{H}} = (h \mid H\bar{e}')_{\mathscr{H}} = \langle h, e'\rangle, \tag{6.1}$$

so, by Proposition 6, λH is the kernel of $\lambda\mathscr{H}$. □

Proposition 12. *If the Hilbert subspaces* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *of* E *have* H_1 *and* H_2 *as kernels, the kernel of* $\mathscr{H}_1 + \mathscr{H}_2$ *is* $H_1 + H_2$.

Proof. Let us use the notation of page 13. The element $(H_1\bar{e}', H_2\bar{e}')$ of $\mathscr{H}_1 \oplus \mathscr{H}_2$, for $e' \in E'$, is orthogonal to the kernel $\mathscr{N}$ of the map Φ. Indeed, let $(k_1, k_2) \in \mathscr{N}$, i.e. such that $k_1 + k_2 = 0$. Then

$$\begin{aligned}&((k_1, k_2) \mid (H_1\bar{e}', H_2\bar{e}'))_{\mathscr{H}_1\oplus\mathscr{H}_2}\\ &= (k_1 \mid H_1\bar{e}')_{\mathscr{H}_1} + (k_2 \mid H_2\bar{e}')_{\mathscr{H}_2} = \langle k_1, e'\rangle + \langle k_2, e'\rangle\\ &= \langle k_1 + k_2, e'\rangle = 0.\end{aligned} \tag{6.2}$$

But the scalar product of two vectors in $\mathscr{H}_2 \oplus \mathscr{H}_2$, if one of them is orthogonal to $\mathscr{N}$, is preserved by Φ, according to (2.11). So, for any $h = h_1 + h_2 \in \mathscr{H}_1 + \mathscr{H}_2$ and $e' \in E'$:

$$\begin{aligned}(h \mid (H_1 + H_2)\bar{e}')_{\mathscr{H}_1+\mathscr{H}_2} &= (h_1 + h_2 \mid H_1\bar{e}' + H_2\bar{e}')_{\mathscr{H}_1+\mathscr{H}_2}\\ &= ((h_1, h_2) \mid (H_1\bar{e}', H_2\bar{e}'))_{\mathscr{H}_1\oplus\mathscr{H}_2}\\ &= (h_1 \mid H_1\bar{e}')_{\mathscr{H}_1} + (h_2 \mid H_2\bar{e}')_{\mathscr{H}_2} = \langle h_1, e'\rangle + \langle h_2, e'\rangle\\ &= \langle h_1 + h_2, e'\rangle = \langle h, e'\rangle,\end{aligned} \tag{6.3}$$

so, by Proposition 6, $H_1 + H_2$ is the kernel of $\mathscr{H}_1 + \mathscr{H}_2$. □

Proposition 13. *Let* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *be two Hilbert subspaces of* E*, and* H_1 *and* H_2 *their kernels. The relationship* $\mathscr{H}_1 \leqq \mathscr{H}_2$ *is equivalent to the relationship* $H_1 \leqq H_2$.

Proof. 1°) Assume that $\mathscr{H}_1 \leqq \mathscr{H}_2$. Then, for any $e' \in E'$, $H_1\bar{e}' \in \mathscr{H}_1$, so $H_1\bar{e}' \in \mathscr{H}_2$, and, according to (4.7):

$$\begin{aligned}\langle H_1\bar{e}', e'\rangle^{1/2} &= \left\|H_1\bar{e}'\right\|_{\mathscr{H}_1} \geqq \left\|H_1\bar{e}'\right\|_{\mathscr{H}_2}\\ &= \sup_{f'\in E'} \frac{\left|\langle H_1\bar{e}', f'\rangle\right|}{(H_2\bar{f}', f')^{1/2}} \geqq \frac{\left|\langle H_1\bar{e}', e'\rangle\right|}{\langle H_2\bar{e}', e'\rangle^{1/2}}\end{aligned} \tag{6.4}$$

so finally

$$\langle H_1\bar{e}', e'\rangle^{1/2} \leqq \langle H_2\bar{e}', e'\rangle^{1/2}, \qquad \text{or } H_1 \leqq H_2. \tag{6.5}$$

2°) Conversely, let us assume that $H_1 \leqq H_2$. Then, for $h \in E$:

$$\sup_{e' \in E'} \frac{\left|\langle h, e'\rangle\right|}{\langle H_1 \bar{e}', e'\rangle^{1/2}} \geqq \sup_{e' \in E'} \frac{\left|\langle h, e'\rangle\right|}{\langle H_2 \bar{e}', e'\rangle^{1/2}}; \tag{6.6}$$

according to Proposition 8, $h \in \mathscr{H}_1$ implies that the left-hand side of (6.6) is finite, so the right-hand side is finite and $h \in \mathscr{H}_2$; moreover, $\|h\|_{\mathscr{H}_1} \geqq \|h\|_{\mathscr{H}_2}$, and as a result $\mathscr{H}_1 \leqq \mathscr{H}_2$. This finishes the proof of Proposition 13, and hence proves the Theorem.

□

§7. Consequences of the isomorphism

Proposition 14. *Let $\mathscr{H}$ and $\mathscr{H}_1$ be two Hilbert subspaces of E. For the existence of a Hilbert subspace $\mathscr{H}_2$ such that $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$, it is necessary and sufficient that $\mathscr{H} \geqq \mathscr{H}_1$, and $\mathscr{H}_2$ is then unique. We write $\mathscr{H}_2 = \mathscr{H} - \mathscr{H}_1$.*

Proof. Let H and H_1 be the kernels of $\mathscr{H}$ and $\mathscr{H}_1$. The existence of $\mathscr{H}_2$ is equivalent to the existence of a kernel $H_2 \geqq 0$ such that $H = H_1 + H_2$, and then $\mathscr{H}_2$ is unique since its kernel $H_2 = H - H_1$ is unique; but the existence of H_2 is equivalent to the relationship $H - H_1 \geqq 0$ or $H \geqq H_1$, which is itself equivalent to $\mathscr{H} \geqq \mathscr{H}_1$. □

Proposition 15. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert subspaces of E, with kernels H_1 and H_2. For $\mathscr{H}_1 \subset \mathscr{H}_2$ to hold, it is necessary and sufficient that there exists a constant $c \geqq 0$ such that $H_1 \leqq cH_2$.*

Corollary. *For $\mathscr{H}_1 \sim \mathscr{H}_2$ to hold, that is, for $\mathscr{H}_1$ and $\mathscr{H}_2$ to be the same vector subspace of E (with possibly different Hilbert structures), it is necessary and sufficient that there exists a constant $c \geqq 0$ such that $H_1 \leqq cH_2$ and $H_2 \leqq cH_1$.*

This Proposition and its Corollary result from Proposition 2 and its Corollary, by applying the Theorem.

Proposition 16. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert subspaces of E, with kernels H_1 and H_2. For $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$ to hold, it is necessary and sufficient for H_1 and H_2 to be alien, that is to say that any kernel $K \geqq 0$ satisfying $K \leqq H_1$ and $K \leqq H_2$ is the zero kernel.*

This results from Proposition 3 and the Theorem.

Proposition 17. *Let $\mathscr{H}$ and $\mathscr{H}_1$ be two Hilbert subspaces of E, with kernels H and H_1. For $\mathscr{H}_1$ to be a subspace of $\mathscr{H}$ with the induced Hilbert structure, it is necessary and sufficient that $H - H_1 \geqq 0$ and that H_1 and $H - H_1$ are alien; then the space $\mathscr{H}_2$ such that $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$ is the orthogonal complement of $\mathscr{H}_1$ in $\mathscr{H}$ with the induced Hilbert structure.*

Proof. 1°) Let us assume that $\mathscr{H}_1 \subset \mathscr{H}$ and that $\mathscr{H}_1$ has the induced Hilbert structure. If then $\mathscr{H}_2$ is the orthogonal complement of $\mathscr{H}_1$ in $\mathscr{H}$, we have $\mathscr{H}_1 + \mathscr{H}_2 = \mathscr{H}$ (pages 13-14). If H_2 is the kernel of $\mathscr{H}_2$, we have $H_1 + H_2 = H$ or $H_2 = H - H_1 \geqq 0$; but $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$, so H_1 and $H - H_1$ are alien (Proposition 16).

2°) Conversely, let us assume that $H - H_1 \geqq 0$ and that H_1 and $H_2 = H - H_1$ are alien. Then, if $\mathscr{H}_2$ is the Hilbert subspace with kernel H_2, we will have $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$ and $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$; so $\mathscr{H}_1 \subset \mathscr{H}$ with the induced Hilbert structure.

□

Proposition 18. 1°) *Let $(\mathscr{H}_i)_{i\in I}$ be a descending filtration of Hilbert subspaces of E, with kernels H_i.*

Then $\inf_{i\in I} \mathscr{H}_i = \mathscr{H}$ exists, and its kernel is $\inf_{i\in I} H_i = H$, which is also $\lim_i H_i$, in $\mathscr{L}(\bar{E}'; E)$ equipped with the topology of weak pointwise convergence. $\mathscr{H}$ is the subspace of $\bigcap_{i\in I} \mathscr{H}_i$ consisting of h such that

$$\sup_{i\in I} \|h\|_{\mathscr{H}_i} < \infty, \text{ and then}$$

$$\text{(7.1)} \qquad \begin{cases} \|h\|_{\mathscr{H}} = \sup_{i\in I} \|h\|_{\mathscr{H}_i} = \lim_i \|h\|_{\mathscr{H}_i} \\ (h \mid k)_{\mathscr{H}} = \lim_{i\in I} (h \mid k)_{\mathscr{H}_i} \end{cases}$$

2°) *Let $(\mathscr{H}_i)_{i\in I}$ be an ascending filtration of Hilbert subspaces of E, with kernels H_i. For it to be bounded from above in Hilb(E), it is necessary and sufficient that $\sup_{i\in I} \langle H_i \bar{e}', e'\rangle < +\infty$ for every $e' \in E'$. In this case, $\sup_{i\in I} \mathscr{H}_i = \mathscr{H}$ exists, and its kernel is $\sup_{i\in I} H_i = H$, which is also $\lim_i H_i$ with respect to the topology of weak pointwise convergence. $\mathscr{H}$ is the Q-completion in E of the pre-Hilbert subspace $\bigcup_{i\in I} \mathscr{H}_i$ equipped with the structure*

$$\text{(7.2)} \qquad \begin{cases} \|h\| = \inf_{i\in I} \|h\|_{\mathscr{H}_i} = \lim_i \|h\|_{\mathscr{H}_i} \\ (h \mid k) = \lim_i (h \mid k)_{\mathscr{H}_i}. \end{cases}$$

Proof. 1°) Let us first start with a descending filtration. I is an ordered index set, directed upwards; for $j \geqq i$, we have $\mathscr{H}_j \leqq \mathscr{H}_i$ and $H_j \leqq H_i$. Limits are taken in the upwards direction in I. For every $e' \in E'$, $\lim_i \langle H_i \bar{e}', e'\rangle$ exists and is equal to $\inf_{i\in I} \langle H_i \bar{e}', e'\rangle$; then (3.7) shows that $\lim_i \langle H_i \bar{f}', e'\rangle$ exists for every e', f' in E'; let us call it $A(e', f')$. A is a non-negative Hermitian form on $E' \times E'$. For any $i \in I$, we have $A(e', e') \leqq \langle H_i \bar{e}', e'\rangle$, so, by the Corollary of Proposition 10, A is separately weakly continuous, and there exists (Proposition 5) a kernel $H \geqq 0$ such that $A(e', f') = \langle H \bar{f}', e'\rangle$. H is the limit of H_i in $\mathscr{L}(\bar{E}'; E)$ (with respect to weak pointwise convergence) and it is trivially the infimum of H_i in $\mathscr{L}^+(E)$. The correspondence between the order structures of Hilb(E) and $\mathscr{L}^+(E)$ shows that the $\mathscr{H}_i$ necessarily have a infimum $\mathscr{H}$ in Hilb(E), and that the kernel of $\mathscr{H}$ is H. With the notations of the Remark that follows Proposition 8, we have

$$\text{(7.3)} \qquad \begin{aligned} \|h\|_{\mathscr{H}} &= \sup_{e'\in E'} \frac{|\langle h, e'\rangle|}{\langle H\bar{e}', e'\rangle^{1/2}} = \sup_{e'\in E'} \left(\sup_{i\in I} \frac{|\langle h, e'\rangle|}{\langle H_i, \bar{e}', e'\rangle^{1/2}} \right) \\ &= \sup_{i\in I} \left(\sup_{e'\in E'} \frac{|\langle h, e'\rangle|}{\langle H_i \bar{e}', e\rangle^{1/2}} \right) = \sup_{i\in I} \|h\|_{\mathscr{H}_i}. \end{aligned}$$

So $\mathscr{H}$ is the subspace of $\bigcap_{i\in I} \mathscr{H}_i$ formed of the h such that $\sup_{i\in I} \|h\|_{\mathscr{H}_i} < \infty$, and we have the first relationship in (7.1); the second results from (2.19b).

2°) Let us now take an ascending filtration; I has the same properties but $j \geqq i$ entails $\mathscr{H}_j \geqq \mathscr{H}_i$ and $H_j \geqq H_i$. Let us suppose that the $\mathscr{H}_i$ are bounded from above by a Hilbert subspace $\mathscr{K}$ with kernel K; then

$$\sup_{i\in I} \langle H_i \bar{e}', e' \rangle \leqq \langle K\bar{e}', e' \rangle < +\infty.$$

Conversely, let us suppose that this condition is realised: $\sup_{i\in I} \langle H_i \bar{e}', e' \rangle < +\infty$. Then $\lim_i \langle H_i \bar{e}', e' \rangle$ exists and is equal to $\sup_{i\in I} \langle H_i \bar{e}', e' \rangle$; (3.7) shows that $\lim_i \langle H_i \bar{f}', e' \rangle$ exists; let us call it $A(e', f')$. A is a non-negative Hermitian form on $E' \times E'$ (a priori not necessarily separately weakly continuous). By the remark following Proposition 5, A comes from a kernel H relative to E'^*, defining a Hilbert subspace $\mathscr{H}$ of E'^*. $\mathscr{H}$ bounds all the $\mathscr{H}_i$ from above, so $\mathscr{H} \cap E$ (which is a Hilbert space by Proposition 0) bounds them from above too: the ascending filtration is bounded from above. Moreover, H is a weak pointwise limit and the supremum of the H_i in $\mathscr{L}(\bar{E}'; E'^*)$, so $\mathscr{H}$ is the supremum of the $\mathscr{H}_i$ in $\mathrm{Hilb}(E'^*)$. As $\mathscr{H} \cap E$ bounds the $\mathscr{H}_i$ from above and $\mathscr{H}$ from below, we necessarily have $\mathscr{H} \cap E = \mathscr{H}$, or $\mathscr{H} \subset E$, and $H \in \mathscr{L}(\bar{E}'; E)$. Let us determine this $\mathscr{H}$. Firstly, $\mathscr{H} \supset \mathscr{H}_i$, so $\mathscr{H} \supset \bigcup_{i\in I} \mathscr{H}_i$, which is a vector space because we have a filtration. On $\bigcup_{i\in I} \mathscr{H}_i$, $h \mapsto \inf_{i\in I} \|h\|_{\mathscr{H}_i} = \lim_i \|h\|_{\mathscr{H}_i}$ is a semi-norm; Equation (2.19b) then shows that we can define a non-negative Hermitian scalar product $(h \mid k) = \lim_i (h \mid k)_{\mathscr{H}_i}$ on it, and that $\|h\| = (h \mid h)$, so $\bigcup_{i\in I} \mathscr{H}_i$ is pre-Hilbert. Moreover, $\|h\|_{\mathscr{H}_i} \geqq \|h\|_{\mathscr{H}}$ so $\inf_{i\in I} \|h\|_{\mathscr{H}_i} \geqq \|h\|_{\mathscr{H}}$. The inclusion $\bigcup_{i\in I} \mathscr{H}_i \to \mathscr{H}$ is thus continuous, and a fortiori the inclusion $\bigcup_{i\in I} \mathscr{H}_i \to E$ is also continuous (so $\bigcup_{i\in I} \mathscr{H}_i$ is Hausdorff). Thus $\bigcup_{i\in I} \mathscr{H}_i$ is a pre-Hilbert subspace of E, which is evidently the smallest pre-Hilbert subspace of E that bounds the $\mathscr{H}_i$ from above. By Proposition 1b (or the remark on page 17), there is a supremum of the $\mathscr{H}_i$; namely the Q-completion of $\bigcup_{i\in I} \mathscr{H}_i$ in E, which is thus $\mathscr{H}$.

□

We have shown, as a by-product, the following result:

Corollary. *If the A_i are separately weakly continuous non-negative Hermitian forms on $E' \times E'$, if they constitute an ascending filtration, and if, for every $e' \in E'$, $\sup_{i\in I} A_i(e', e') < +\infty$, they have a pointwise limit on $E' \times E'$, which is again separately weakly continuous.*

Remark 1. $\mathrm{Hilb}(E)$ and $\mathscr{L}^+(E)$ *are not lattices. There exists an infimum of a descending filtration* but not an infimum of any two elements. (In order to see this, it suffices to remark that, for $E = \mathbb{C}^2$, we have $E' = \mathbb{C}^2$, and that there does not exist, in general, an infimum of two non-negative Hermitian forms on $\mathbb{C}^2$). Likewise, there exists a supremum of an upper-bounded ascending filtration, but not, in general, a supremum of two elements. (A set with two elements is bounded from above by their sum).

Remark 2. An ascending filtration of $\mathrm{Hilb}(E)$ is not necessarily bounded from above. It is the case if I is the set of non-negative reals and $\mathscr{H}_t = t\mathscr{K}$, where $\mathscr{K} \neq \{0\}$ is fixed and $t \geqq 0$ is real.

Remark 3. In the case of a descending filtration, $\mathscr{H} = \inf\limits_{i\in I} \mathscr{H}_i$ can be different from $\bigcap\limits_{i\in I} \mathscr{H}_i$. This is the case if I is the set of real numbers strictly greater than 0, and $\mathscr{H}_t = \frac{1}{t}\mathscr{K}$, where $\mathscr{K} \neq \{0\}$ is fixed and $t > 0$. Then $\bigcap\limits_{t>0} \mathscr{H}_t = \mathscr{K}$, however $\inf\limits_{t>0} \mathscr{H}_t = \{0\}$.

Remark 4. In the case of an ascending filtration bounded from above, it could be that $\bigcup\limits_{i\in I} \mathscr{H}_i$, a pre-Hilbert subspace of E, does not have a completion in E. Let, for example, $\mathscr{H}_0$ be a pre-Hilbert subspace of E, but not having a completion in E (see counterexample (1.5)). If the $\mathscr{H}_i$ are the subspaces of $\mathscr{H}_0$ with finite dimension, equipped with the induced structure, they are Hilbert subspaces of E. They constitute an ascending filtration, bounded from above by the Q-completion $\mathscr{H}$ of $\mathscr{H}_0$ in E. Then $\bigcup\limits_{i\in I} \mathscr{H}_i = \mathscr{H}_0$ which does not have a completion in E, and $\sup\limits_{i\in I} \mathscr{H}_i = \mathscr{H}$.

Infinite sums of Hilbert subspaces of E.

Let $(\mathscr{H}_i)_{i\in I}$ be a family of Hilbert subspaces of E. We can define their abstract Hilbert sum $\hat{\bigoplus\limits_{i\in I}} \mathscr{H}_i$; an element of this sum is a family $(h_i)_{i\in I}, h_i \in \mathscr{H}_i$, such that $\sum\limits_{i\in I} \|h_i\|^2_{\mathscr{H}_i} < +\infty$, with

$$(7.4) \qquad \left\|(h_i)_{i\in I}\right\|^2_{\hat{\bigoplus}_{i\in I} \mathscr{H}_i} = \sum_{i\in I} \|h_i\|^2_{\mathscr{H}_i}.$$

The elements $(h_i)_{i\in I}$, for which all but a finite number of the h_i are zero, form a dense subspace $\bigoplus\limits_{i\in I} \mathscr{H}_i$ of $\hat{\bigoplus\limits_{i\in I}} \mathscr{H}_i$.

Proposition 19. *The following three properties are equivalent.*

1°) *The linear map Φ_0 from $\bigoplus\limits_{i\in I} \mathscr{H}_i$ into E, defined by*

$$(7.5) \qquad \Phi_0\left((h_i)_{i\in I}\right) = \sum_{i\in I} h_i,$$

is continuous.

2°) *The finite sums $\sum\limits_{i\in J} \mathscr{H}_i$, for finite subsets J of I, are bounded from above in* $\mathrm{Hilb}(E)$.

3°) *For every $e' \in E'$, $\sum\limits_{i\in I} \langle H_i \bar{e}', e'\rangle < +\infty$.*

If these hold, we say that the family $(\mathscr{H}_i)_{i\in I}$ *is summable. Then* Φ_0 *extends to a continuous linear map* Φ *from* $\hat{\bigoplus}_{i\in I} \mathscr{H}_i$ *into* E*, defined again by (7.5) (the condition* $\sum_{i\in I} \|h_i\|^2_{\mathscr{H}_i} < +\infty$ *implies that* $\sum_{i\in I} h_i$ *are summable in* E*). Let* $\mathscr{N}$ *be the kernel of* Φ*;* Φ *factorises into* $\hat{\bigoplus}_{i\in I} \mathscr{H}_i \xrightarrow{\pi} (\hat{\bigoplus}_{i\in I} \mathscr{H}_i)/\mathscr{N} \xrightarrow{\Phi^\bullet} E$*.*

We denote the image $\Phi(\hat{\bigoplus}_{i\in I} \mathscr{H}_i)$ *by* $\sum_{i\in I} \mathscr{H}_i$*, with the Hilbert structure transported by* $\Phi^\bullet$ *from that of* $(\hat{\bigoplus}_{i\in I} \mathscr{H}_i)/\mathscr{N}$*. The elements of* $\sum_{i\in I} \mathscr{H}_i$ *are those that can be expressed as sums* $\sum_{i\in I} h_i$*, summable in* E*, with* $h_i \in \mathscr{H}_i$*,* $\sum_{i\in I} \|h_i\|^2_{\mathscr{H}_i} < \infty$*, and we have*

$$\|h\|^2_{\sum_{i\in I} \mathscr{H}_i} = \inf_{(\sum_{i\in I} h_i)=h} \left(\sum_{i\in I} \|h_i\|^2_{\mathscr{H}_i} \right). \tag{7.6}$$

Moreover, $\sum_{i\in I} \mathscr{H}_i$ *is equal to the supremum of the finite sums* $\sum_{i\in J} \mathscr{H}_i$*, where* J *is a finite subset of* I*; and its kernel is* $\sum_{i\in I} H_i$*, summable with respect to the topology* $\mathscr{L}_s(\bar{E}'; E)$ *of pointwise convergence.*

Proof. Suppose that 1°) holds. Then Φ_0 extends to a continuous linear map Φ from $\hat{\oplus}_{i\in I}\mathscr{H}_i$ into E. If $(h_i)_{i\in I}$ is an element of $\hat{\bigoplus}_{i\in I} \mathscr{H}_i$, it is the limit of the elements $((h_i)_{i\in J}, (0)_{i\in I-J})$, following the filtration of finite subsets J of I; its image under Φ is thus necessarily the limit of $\sum_{i\in J} h_i$ following this filtration, that is to say, the sum $\sum_{i\in I} h_i$, summable in E.

Considering the definition of the quotient norm in $(\hat{\bigoplus}_{i\in I} \mathscr{H}_i)/\mathscr{N}$ and the fact that $\Phi^\bullet$ transports the Hilbert structure, $\sum_{i\in I} \mathscr{H}_i$ defined in the statement has the norm in (7.6). But, if J is a finite subset of I, an element k of $\sum_{i\in J} \mathscr{H}_i$ is of the form $\sum_{i\in J} k_i$, and we have, according to (2.9),

$$\|k\|^2_{\sum_{i\in J} \mathscr{H}_i} = \inf_{(\sum_{i\in J} k_i)=k} \|k_i\|^2_{\mathscr{H}_i}; \tag{7.7}$$

so $\sum_{i\in J} \mathscr{H}_i$ is a subspace of $\sum_{i\in I} \mathscr{H}_i$, with an inclusion of norm $\leqq 1$: the $\sum_{i\in J} \mathscr{H}_i$ is bounded from above by $\sum_{i\in I} \mathscr{H}_i$, and we therefore have 2°).

Let us now suppose that 2°) holds. As the kernel of $\sum_{i\in J} \mathscr{H}_i$ is $\sum_{i\in J} H_i$, Proposition 11 shows that we have 3°).

Finally, suppose that we have 3°). To see that Φ_0 is continuous, we have to show that the image under Φ_0 of the unit ball B_0 of $\bigoplus_{i\in I} \mathscr{H}_i$ is weakly bounded. So let $e' \in E'$. $\Phi_0(B_0)$ is the set of the finite sums $\sum_{i\in J} h_i$, such that $\sum_{i\in J} \|h_i\|^2_{\mathscr{H}_i} \leqq 1$.

For $h \in \Phi_0(B_0)$ and $h = \sum_{i \in J} h_i$, we have

$$
\begin{aligned}
\left|\langle h, e' \rangle\right| &\leqq \sum_{i \in J} \left|\langle h_i, e' \rangle\right| = \sum_{i \in J} \left|(h_i \mid H_i \bar{e}')_{\mathscr{H}_i}\right| \\
&\leqq \left(\sum_{i \in J} \|h_i\|^2_{\mathscr{H}_i} \right)^{1/2} \left(\sum_{i \in J} \langle H_i \bar{e}', e' \rangle \right)^{1/2} \\
&\leqq \left(\sum_{i \in I} \langle H_i \bar{e}', e' \rangle \right)^{1/2} \quad ;
\end{aligned}
\tag{7.8}
$$

so $\Phi_0(B_0)$ is weakly bounded, Φ_0 is continuous, we have $1°$), and the three given conditions are equivalent. Let us suppose that they hold. Proposition 18 says that the series $\sum_{i \in I} H_i$ is summable with respect to the topology of weak pointwise convergence; we have to show here that it is summable with respect to the topology of strong pointwise convergence. But $\sum_{i \in I} \|H_i \bar{e}'\|^2_{\mathscr{H}_i} = \sum_{i \in I} \langle H_i \bar{e}', e' \rangle < +\infty$; so the image under Φ of $(H_i \bar{e}')_{i \in I} \in \oplus \mathscr{H}_i$, in other words, $\sum_{i \in I} H_i \bar{e}'$, is summable in E, which proves our assertion. But Proposition 18 says that $\sum_{i \in I} H_i = \sup_{J \text{ finite} \subset I} (\sum_{i \in J} H_i)$ is the kernel of $\sup_{J \text{ finite} \subset I} (\sum_{i \in J} \mathscr{H}_i)$; if we show that it is also the kernel of $\sum_{i \in i} \mathscr{H}_i$, we will have finished the proof.

The element $(H_i \bar{e}')_{i \in I}$ of $\hat{\bigoplus}_{i \in I} \mathscr{H}_i$ is orthogonal to the kernel $\mathscr{N}$ of Φ; this can be seen from (6.2). We thus have, as in (6.3), for $h \in \sum_{i \in I} h_i \in \sum_{i \in I} \mathscr{H}_i$:

$$
\begin{aligned}
&\langle \sum_{i \in I} h_i, e' \rangle = \sum_{i \in I} \langle h_i, e' \rangle \\
&= \sum_{i \in I} (h_i \mid H_i \bar{e}')_{\mathscr{H}_i} = ((h_i)_{i \in I} \mid (H_i \bar{e}')_{i \in I})_{\hat{\bigoplus}_{i \in I} \mathscr{H}_i} \\
&= \left(\sum_{i \in I} h_i \mid \left(\sum_{i \in I} H_i \right) \bar{e}' \right)_{\sum_{i \in I} \mathscr{H}_i} ,
\end{aligned}
\tag{7.9}
$$

which proves that $\sum_{i \in I} H_i$ is the kernel of $\sum_{i \in I} \mathscr{H}_i$. □

Corollary 1. *Let $\mathscr{H}$ be a Hilbert subspace of E, a direct Hilbert sum of closed (pairwise orthogonal) subspaces $\mathscr{H}_i$, $i \in I$. Then the kernel H of $\mathscr{H}$ is the sum $\sum_i H_i$ of the kernels of $\mathscr{H}_i$, the series being summable with respect to the topology $\mathscr{L}_s(\bar{E}'; E)$ of pointwise convergence.*

Indeed, in this case, $\sum_{i \in I} \mathscr{H}_i = \mathscr{H}$ (here, $\mathscr{N} = \{0\}$ and Φ is an isomorphism from $\hat{\bigoplus}_{i \in I} \mathscr{H}_i$ onto $\mathscr{H}$).

Corollary 2. *Let $\mathscr{H}$ be a Hilbert subspace of E, and let $(e_i)_{i \in I}$ be a Hilbert basis of $\mathscr{H}$. Then the kernel of $\mathscr{H}$ is $\sum_i e_i \otimes \bar{e}_i$, a summable series in $\mathscr{L}_s(\bar{E}'; E)$.*

Proof. Let $e \in E$. The element e' of $\bar{E}$ defines the scalar form $\bar{f}' \mapsto \overline{\langle e, f' \rangle}$ on $\bar{E}'$. Then $e \otimes \bar{e} \in E \otimes \bar{E} \subset E\varepsilon\bar{E}$ is the sesquilinear form $(f', g') \mapsto \langle e, f' \rangle \overline{\langle e, g' \rangle}$ on $E' \times E'$, or the kernel $H_e : \bar{f}' \mapsto \overline{\langle e, f' \rangle} e$.

We can associate to the vector e of E the Hilbert subspace $\mathscr{H}_e = \mathbb{C}e$ of E, equipped with the scalar product:

$$(\alpha e \mid \beta e)_{\mathscr{H}_e} = \alpha\bar{\beta}. \tag{7.10}$$

Then $H_e = e \otimes \bar{e}$ is the kernel of $\mathscr{H}_e$, because, for any $h = \alpha e \in \mathscr{H}_e$ and $f' \in E'$, we have

$$\langle h, f' \rangle = \alpha \langle e, f' \rangle = (\alpha e \mid \overline{\langle e, f' \rangle} e)_{\mathscr{H}_e} = (h \mid H_e \bar{f}')_{\mathscr{H}_e}. \tag{7.11}$$

Then, if the e_i form a Hilbert basis of $\mathscr{H}$, $\mathscr{H}$ is the Hilbert direct sum of the subspaces $\mathscr{H}_{e_i}$, and it suffices to apply Corollary 1 to obtain the result. □

Corollary 3. *(Orthogonal decomposition of a non-negative kernel). Every kernel $H \geqq 0$ relative to E admits a decomposition into a sum $\sum_{i \in I} e_i \otimes \bar{e}_i$, $e_i \in E$, the series being summable with respect to the topology of pointwise convergence; I is countable if E' is weakly separable. (Apply the preceding Corollary, and the Corollary of Proposition 7).*

Corollary 4. *Let $(\mathscr{H}_i)_{i \in I}$ be a family of Hilbert subspaces of E, with kernels H_i. For the existence of a Hilbert subspace of E that admits the $\mathscr{H}_i$ as closed orthogonal subspaces with the induced Hilbert structures, the following two conditions are necessary and sufficient:*

1°) *For any $e' \in E'$, $\sum_{i \in I} \langle H_i \bar{e}', e' \rangle < +\infty$;*

2°) *The system of $\mathscr{H}_i$ is Hilbert-free in E, in the sense that, if $h_i \in \mathscr{H}_i$ satisfy $\sum_{i \in I} \|h_i\|^2_{\mathscr{H}_i} < +\infty$ and $\sum_{i \in I} h_i = 0$ (the series being summable in E), then all h_i are zero.*

Proof. The existence of such a space is exactly implied by the family $(\mathscr{H}_i)_{i \in I}$ being summable, i.e. the first condition, and that the map Φ of Proposition 19 is an isomorphism from $\hat{\bigoplus}_{i \in I} \mathscr{H}_i$ onto $\sum_{i \in I} \mathscr{H}_i$ (with kernel $\mathscr{N} = \{0\}$), i.e. the second condition. □

Corollary 5. *Let $(e_i)_{i \in I}$ be a family of elements of E. For the existence of a Hilbert subspace of E that admits them as a Hilbert basis, the following two conditions are necessary and sufficient:*

1°) *For any $f' \in E'$, $\sum_{i \in I} |\langle e_i, f' \rangle|^2 < +\infty$.*

2°) *The system of the e_i is Hilbert-free in E, in the sense that, if some $c_i \in \mathbb{C}$ satisfying $\sum_{i \in I} |c_i|^2 < +\infty$ and $\sum_{i \in I} c_i e_i = 0$ (summable series in E), then all c_i are zero.*

It suffices to apply Corollary 4 to the $\mathscr{H}_{e_i}$ (in the notation of Corollary 2).

Integral of Hilbert subspaces of E.

Let Z be a locally compact space equipped with a measure $\mu \geqq 0$. Let $\zeta \to \mathscr{H}(\zeta)$ be a map from Z into $\mathrm{Hilb}(E)$; we will say that this map is μ-integrable (or that the family $(\mathscr{H}(\zeta))_{\zeta \in Z}$ of Hilbert subspaces is μ-integrable) if, for every $e' \in E'$, the function $\zeta \to \langle H(\zeta)\bar{e}', e'\rangle$ (where $H(\zeta)$ is the kernel of $\mathscr{H}(\zeta)$) is μ-integrable. In other words, $\zeta \to H(\zeta)$ is weakly μ-integrable in $\mathscr{L}(\bar{E}'; E)$ equipped with the topology of weak pointwise convergence.

Proposition 20. *Let $\zeta \to \mathscr{H}(\zeta)$ be a μ-integrable family of Hilbert subspaces of E, and let us suppose that the dual E' is weakly separable. Let us say that a field of vectors $\zeta \mapsto h(\zeta) \in \mathscr{H}(\zeta)$ is μ-measurable if the function $\zeta \to h(\zeta)$ on Z with values in E is weakly μ-measurable; these fields define on $\mathscr{H}(\zeta)$ a structure S of a μ-measurable field of Hilbert spaces*[39]. *Let us denote by $\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta)$ the space of classes (modulo equality μ-almost everywhere) of fields of μ-measurable vectors $\zeta \to h(\zeta)$ such that $\int_Z \|h(\zeta)\|^2_{\mathscr{H}(\zeta)} d\mu(\zeta) < +\infty$ with*

$$(7.12) \qquad \left\|h(\zeta)_{\zeta \in Z}\right\|^2_{\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta)} = \int_Z \left\|h(\zeta)\right\|^2_{\mathscr{H}(\zeta)} d\mu(\zeta).$$

There exists a continuous linear map Φ from $\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta)$ into E'^* (equipped with the topology $\sigma(E'^*, E')$) defined by:

$$(7.13) \qquad \Phi(h(\zeta)_{\zeta \in Z}) = \int_Z h(\zeta)d\mu(\zeta) \in E'^*,$$

the right-hand side being the weak integral of a weakly integrable function. Let $\mathscr{N}$ be the kernel of Φ. Then Φ factorises into

$$\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta) \xrightarrow{\pi} \left(\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta)\right)/\mathscr{N} \xrightarrow{\Phi^\bullet} E.$$

We denote by $\mathscr{H} = \int_Z \mathscr{H}(\zeta)d\mu(\zeta)$ the Hilbert subspace of E'^* that is transported by $\Phi^\bullet$ from the quotient $(\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta))/\mathscr{N}$.

It is set of the $h \in E'^*$ that can be written as

$$\int_Z h(\zeta)d\mu(\zeta), \qquad (h(\zeta))_{\zeta \in Z} \in \int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta),$$

with the norm

$$(7.14) \qquad \|h\|^2_{\int_Z \mathscr{H}(\zeta)d\mu(\zeta)} = \inf_{\int_Z h(\zeta)d\mu(\zeta) = h} \int_Z \left\|h(\zeta)\right\|^2_{\mathscr{H}(\zeta)} d\mu(\zeta).$$

The kernel of $\mathscr{H} = \int_Z \mathscr{H}(\zeta)d\mu(\zeta)$ relative to E'^* is

$$H = \int_Z H(\zeta)d\mu(\zeta) \in \mathscr{L}(\bar{E}'; E'^*)$$

(weak integral of $\zeta \to H(\zeta)$, a weakly integrable function with values in $\mathscr{L}(\bar{E}'; E)$ equipped with the topology of weak pointwise convergence).

[39] Here, we follow the notations and terminology of Dixmier [1].

Proof. Let e'_n, $n = 1, 2, ...$ be a weakly total sequence in E'. Let us consider the fields of vectors $\zeta \to h_n(\zeta) = H(\zeta)\bar{e}'_n$. For any ζ, the $H(\zeta)\bar{e}'_n$ form a weakly total, and therefore strongly total, sequence in $H(\zeta)(\bar{E}') \subset \mathscr{H}(\zeta)$, and hence in $\mathscr{H}(\zeta)$ itself.

Moreover, the function $\zeta \to (h_m(\zeta) \mid h_n(\zeta))_{\mathscr{H}(\zeta)} = \langle H(\zeta)\bar{e}'_m, \bar{e}'_n\rangle$ is measurable by the hypothesis made on the kernels $H(\zeta)$.

So[40] there exists a unique measurable field structure S of Hilbert spaces on the family of the $\mathscr{H}(\zeta)$, for which the $\zeta \to h_n(\zeta)$ are measurable fields. If $\zeta \to h(\zeta)$ is a measurable field with respect to this structure S, it belongs to the family generated by the $\zeta \to \phi(\zeta)h_n(\zeta)$, where ϕ is a complex measurable function on Z. Then the function $\zeta \to \langle h(\zeta), e'\rangle$ belongs to the family generated by the functions $\zeta \to \langle \phi(\zeta)h_n(\zeta), e'\rangle = \phi(\zeta)\langle H(\zeta)\bar{e}'_n, e'\rangle$; these are measurable by the hypothesis made on the kernels $H(\zeta)$, so $\zeta \to \langle h(\zeta), e'\rangle$ is measurable, and $\zeta \to h(\zeta)$ is weakly measurable on Z with values in E. Conversely, if this is the case, $\zeta \to (h(\zeta) \mid h_n(\zeta))_{\mathscr{H}(\zeta)} = \langle h(\zeta), e'_n\rangle$ is measurable, so $\zeta \to h(\zeta)$ is a measurable field with respect to the structure S. The structure S is therefore indeed what we described in the statement, for which the measurable fields are the maps $\zeta \to h(\zeta) \in \mathscr{H}(\zeta)$, weakly μ-measurable on Z with values in E (in particular, S is independent of the choice of the e'_n).

Let us show that (7.13) makes sense. If $\zeta \to h(\zeta)$ is in $\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta)$, it is a measurable field with respect to the structure S, so weakly measurable with values in E. To show that this function with values in E is weakly integrable, and hence that (7.13) makes sense, it suffices to show that $\zeta \to |\langle h(\zeta), e'\rangle|$ is integrable. But

$$\left|\langle h(\zeta), e'\rangle\right| = \left|(h(\zeta) \mid H(\zeta)\bar{e}')_{\mathscr{H}(\zeta)}\right| \leqq \left\|h(\zeta)\right\|_{\mathscr{H}(\zeta)} \langle H(\zeta)\bar{e}', e'\rangle^{1/2} \tag{7.15}$$

an integrable function as the product of two functions in L^2; moreover:

$$\begin{aligned}
&\left|\left\langle \int_Z h(\zeta)d\mu(\zeta), e'\right\rangle\right| = \left|\int_Z \langle h(\zeta), e'\rangle\, d\mu(\zeta)\right| \\
&\leqq \int_Z \left|\langle h(\zeta), e'\rangle\right| d\mu(\zeta) = \int_Z \left|(h(\zeta) \mid H(\zeta)\bar{e}')_{\mathscr{H}(\zeta)}\right| d\mu(\zeta) \\
&\leqq \left(\int_Z \left\|h(\zeta)\right\|^2_{\mathscr{H}(\zeta)} d\mu(\zeta)\right)^{1/2} \left(\int_Z \langle H(\zeta)\bar{e}', e'\rangle\, d\mu(\zeta)\right)^{1/2};
\end{aligned} \tag{7.16}$$

so Φ is continuous from $\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta)$ into E'^{*}, since the image of the unit ball is bounded.

Then we can describe the space $\int_Z \mathscr{H}(\zeta)d\mu(\zeta) = \mathscr{H} \subset E$ according to the statement, and its norm is (7.14). Let us write $H = \int_Z H(\zeta)d\mu(\zeta)$, a weak integral of $\zeta \to H(\zeta)$ with values in $\mathscr{L}(\bar{E}'; E)$ equipped with the topology of weak pointwise convergence; H is a linear map from $\bar{E}'$ into E'^{*}, with

$$\langle H\bar{f}', e'\rangle = \int_Z \langle H(\zeta)\bar{f}', e'\rangle d\mu(\zeta). \tag{7.17}$$

Let us show that H is the kernel of $\mathscr{H} \subset E'^{*}$. As in Propositions 12 and 19, we see immediately that the field $\zeta \to H(\zeta)\bar{e}'$ is orthogonal to $\mathscr{N}$ in $\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta)$.

[40] Dixmier [1], Proposition 4, page 144.

In the same way again, we deduce from this that, for $h = \int_Z h(\zeta)d\mu(\zeta) \in \mathscr{H}$:

$$
\begin{aligned}
\langle h, e' \rangle = \langle \int_Z h(\zeta)d\mu(\zeta), e' \rangle &= \int_Z \langle h(\zeta), e' \rangle d\mu(\zeta) \\
&= \int_Z (h(\zeta) \mid H(\zeta)\bar{e}')_{\mathscr{H}(\zeta)} d\mu(\zeta) \\
&= (h(\zeta)_{\zeta \in Z} \mid (H(\zeta)\bar{e}')_{\zeta \in Z})_{\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta)} \\
&= (\int_Z h(\zeta)d\mu(\zeta) \mid \int_Z H(\zeta)\bar{e}' d\mu(\zeta))_{\mathscr{H}} \\
&= (h \mid H\bar{e}')_{\mathscr{H}},
\end{aligned}
\tag{7.18}
$$

which, by (4.2), shows that H is the kernel of $\mathscr{H}$. □

Remark. We will often have to show that $\mathscr{H} \subset E$. For that, it will suffice (Proposition 0) to show that $H(\bar{E}') \subset E$, in other words, that the weak integral $\int_Z H(\zeta)d\mu(\zeta) \in \mathscr{L}(\bar{E}'; E'^*)$ is an element of $\mathscr{L}(\bar{E}'; E)$. We have many criteria for that: not only all those concerning weak integrals in general[41], but also, for example, Propositions 18, 2°), and 19.

§8. Effect of a continuous linear map

Let E and F be locally convex, quasi-complete Hausdorff spaces, and u a weakly continuous linear map from E into F[42a]. Let $\mathscr{H}$ be a Hilbert subspace of E; let us denote by $\mathscr{N} = \mathscr{H} \cap u^{-1}(\{0\})$ the kernel[42b] of the restriction of u to $\mathscr{H}$. Then this restriction factorises into $\mathscr{H} \xrightarrow{\pi} \mathscr{H}/\mathscr{N} \xrightarrow{u} F$, where u is an injective continuous linear map, so it is a bijection from $\mathscr{H}/\mathscr{N}$ onto the image $u(\mathscr{H})$ of $\mathscr{H}$ under u. We will then say that $u(\mathscr{H})$ is the Hilbert subspace of F, obtained by transporting the Hilbert structure of $\mathscr{H}/\mathscr{N}$ onto $u(\mathscr{H})$ under the bijection u.

We thus see that the norm of $u(\mathscr{H})$ is simply defined by

$$
\|k\|_{u(\mathscr{H})} = \inf_{h \in \mathscr{H}, u(h)=k} \|h\|_{\mathscr{H}} . \tag{8.1}
$$

We can also say that $u(\mathscr{H})$ is the smallest Hilbert subspace of F (with respect to the order relation $\leqq$) such that u is a continuous linear map of norm $\leqq 1$ from $\mathscr{H}$ into this space. Indeed, if $\mathscr{M}$ is a Hilbert subspace of F containing $u(\mathscr{H})$, and such that $u : \mathscr{H} \to \mathscr{M}$ is of norm $\leqq 1$, then $u : \mathscr{H}/\mathscr{N} \to \mathscr{M}$ is also of norm $\leqq 1$, so the inclusion of $u(\mathscr{H})$ in $\mathscr{M}$ is of norm $\leqq 1$, and $u(\mathscr{H}) \leqq \mathscr{M}$. We have already seen a diverse set of examples: even in the definition of $\mathscr{H}_1 + \mathscr{H}_2$ in §2, 2°), Φ is a continuous linear map from $\mathscr{H}_1 \oplus \mathscr{H}_2$ into E, and $\mathscr{H}_1 + \mathscr{H}_2$ is nothing but the image $\Phi(\mathscr{H}_1 \oplus \mathscr{H}_2)$ in the above sense. It is the same in Propositions 19 and 20: $\sum_{i \in I} \mathscr{H}_i = \Phi(\hat{\bigoplus}_{i \in I} \mathscr{H}_i)$, and

$$
\int \mathscr{H}(\zeta)d\mu(\zeta) = \Phi(\int_Z^{\oplus} \mathscr{H}(\zeta)d\mu(\zeta)).
$$

[41] See Bourbaki [3], Chapter VI, §1.

[42a] Let us recall that, if E is metrisable, a weakly continuous linear map from E into F is continuous with respect to the initial topologies (Bourbaki [1], Chapter IV, §4, n°2, Corollary of Proposition 7). In particular, the restriction of u to $\mathscr{H}$ is continuous from $\mathscr{H}$ into F.

[42b] Kernel here means the pre-image of $\{0\}$, and has nothing to do with the kernels associated to Hilbert subspaces in this article!

Moreover, the space $\lambda\mathscr{H}$ with $\lambda \geqq 0$, defined in §2, 1°), is nothing but the image of $\mathscr{H}$ by the homothety $e \mapsto \sqrt{\lambda}e$ from E into E.

Another important case was implicitly encountered many times. Let us suppose that E is a subspace of F, equipped with a topology such that the inclusion u of E in F is weakly continuous. Then every Hilbert subspace of E is a fortiori a Hilbert subspace of F, which amounts to identifying it with its image under u. For some Hilbert subspaces of E, multiplication by non-negative scalars, addition and the order relation in E and in F are identical; if $\mathscr{H} \geqq \mathscr{H}_1$, the difference $\mathscr{H} - \mathscr{H}_1$ is the same in E or in F (and the same even in $\mathscr{H}$).

If now u is weakly continuous and antilinear from E into F, we will define $u(\mathscr{H})$ with the "anti-transported" Hilbert structure from $\mathscr{H}/\mathscr{N}$ on it by u;

$$(8.1b) \qquad (u(\alpha) \mid u(\beta))_{u(\mathscr{H})} = (\beta \mid \alpha)_{\mathscr{H}/\mathscr{N}}, \qquad \alpha \in \mathscr{H}/\mathscr{N}, \beta \in \mathscr{H}/\mathscr{N}$$

so that we again have (8.1).

Regardless of whether u is linear or antilinear, let $\mathscr{K}$ be the orthogonal complement of $\mathscr{N}$ in $\mathscr{H}$. Then the restriction of u to $\mathscr{K}$ is an isomorphism (or an anti-isomorphism), and $u(\mathscr{H})$ has the structure transported (or anti-transported) from that of $\mathscr{K}$ by u. The scalar product of two vectors of $\mathscr{H}$ is equal to (or the conjugate of) that of their images by u in $u(\mathscr{H})$ each time one of the two is in $\mathscr{K}$. We deduce that the closed (resp. open) unit ball of $u(\mathscr{H})$ is exactly the image under u of the closed (resp. open) unit ball of $\mathscr{H}$. Finally, if E, F and G are three locally convex quasi-complete vector spaces, if u (resp. v) is a weakly continuous linear or antilinear map from E into F (resp. from F to G), and if $\mathscr{H}$ is a Hilbert subspace of E, we have, according to (8.1):

$$(8.1c) \qquad (v \circ u)(\mathscr{H}) = v(u(\mathscr{H})).$$

Proposition 21. *If u is a weakly continuous linear or antilinear map from E into F, and if $\mathscr{H}$ is a Hilbert subspace of E with kernel H, the kernel of $u(\mathscr{H})$ with respect to F is $u(H) = uHu^*$: $\bar{F}' \overset{u^*}{\to} \bar{E}' \overset{H}{\to} E \overset{u}{\to} F$. Moreover, the set of the $Hu^*\bar{f}'$, $f' \in F'$, is a dense subspace of $\mathscr{K}$, the orthogonal complement of $\mathscr{N}$ in $\mathscr{H}$.*

Proof. First, let u be linear. Let us find the orthogonal complement of the set $Hu^*(\bar{F}')$ of the $Hu^*\bar{f}'$ in $\mathscr{H}$. Let h be an element of $\mathscr{H}$:

$$(8.2) \quad (h \mid Hu^*\bar{f}')_{\mathscr{H}} = \langle h, {}^t uf'\rangle_{E,E'} \text{ (because } \overline{u^*\bar{f}'} = {}^t uf') \ = \langle uh, f'\rangle_{F,F'};$$

so h is orthogonal to $Hu^*(\bar{F}')$, if and only if $uh = 0$, or $h \in \mathscr{N}$: $\mathscr{N}$ is the orthogonal complement that we were searching for. Then the orthogonal complement $\mathscr{K}$ of $\mathscr{N}$ is the closure in $\mathscr{H}$ of the set $Hu^*(\bar{F}')$.

Then the scalar product $(h \mid Hu^*\bar{f}')_{\mathscr{H}}$ is always equal to that of the images in $u(\mathscr{H})$:

$$(8.3) \qquad \langle uh, f'\rangle_{F,F'} = (h \mid Hu^*\bar{f}')_{\mathscr{H}} = (uh \mid uHu^*\bar{f}')_{u(\mathscr{H})},$$

which proves, by Proposition 6, that uHu^* is the kernel of $u(\mathscr{H})$ in F.

If now u is antilinear, we will have the same result, but with the modification:

$$(8.4) \qquad \langle uh, f'\rangle_{F,F'} = \overline{(h \mid Hu^*\bar{f}')_{\mathscr{H}}} = (uh \mid uHu^* f')_{u(\mathscr{H})}.$$

□

Remark. Proposition 21 is in fact what served as the definition of the kernel of $\mathscr{H}$ in E. The kernel of $\mathscr{H}$ in $\mathscr{H}$ is the canonical isomorphism A from $\bar{\mathscr{H}}'$ onto $\mathscr{H}$; if j is the inclusion of $\mathscr{H}$ in E, then the kernel of $\mathscr{H} = j(\mathscr{H})$ in E is jAj^* (Equation (4.1)).

Example 1. Let $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ be the Fourier transform[(43)]. We have ${}^t\mathscr{F} = \mathscr{F}$, because of the symmetry of the function $\underline{(x, \xi)} \mapsto \exp(-2i\pi\langle x, \xi\rangle)$. Let us suppose that the reciprocity formula $\mathscr{F}\overline{\mathscr{F}} = \overline{\mathscr{F}}\mathscr{F} = I$ is known (which entails the bijectivity of $\mathscr{F}$); we thus also have $\mathscr{F}\mathscr{F}^* = \mathscr{F}^*\mathscr{F} = I$. The Hilbert subspace L^2 of $\mathscr{S}'$ has the identity I as its kernel (more precisely, the inclusion of $\mathscr{S}$ in $\mathscr{S}'$). The image $\mathscr{F}(L^2)$ thus has $\mathscr{F}I\mathscr{F}^* = \mathscr{F}\mathscr{F}^* = I$ for its kernel: it is L^2 itself. $\mathscr{F}$, being injective, is thus an isometry from L^2 onto L^2: the formula of reciprocity, as it is well-known, results in the formula of Parseval-Plancherel.

Example 2. Let $E = \mathscr{D}'(X)$, the space of distributions on an open subset X of $\mathbb{R}^n$. Let $\mathscr{H}$ be a Hilbert subspace of $\mathscr{D}'(X)$, with kernel $H_{x,\xi} \in \mathscr{D}'(X \times X)$. Let p be an infinitely differentiable complex function on X. The image $p\mathscr{H}$ of $\mathscr{H}$ under multiplication by p has kernel $p(x)\bar{p}(\xi)H_{x,\xi}$, defining the map $\phi \mapsto p(H \cdot \bar{p}\phi)$ from $\mathscr{D}$ into $\mathscr{D}'$.

Indeed, the transpose of multiplication by p is multiplication by p, so its adjoint is multiplication by $\bar{p}$; the kernel of $p\mathscr{H}$ is thus $\phi \underline{\mapsto p}(H \cdot \bar{p}\phi)$ by Proposition 21. This can also be written as $\phi \mapsto p(x) \int_X H_{x,\xi}\overline{p(\xi)}\phi(xi)d\xi$, so it is defined by the distribution $p(x)\overline{p(\xi)}H_{x,\xi}$.

Example 3. Let us go back to the situation of Example 2. Let D^p be a partial derivative of index $p = (p_1, p_2, ..., p_n)$. (No relation between this p and the preceding one!) Its transpose and its adjoint are equal to $(-1)^{|p|}D^p$. The image $D^p\mathscr{H}$ of $\mathscr{H}$ under D^p has kernel $\phi \mapsto (-1)^{|p|}D^p(H \cdot D^p\phi) = (1)^{|p|}D_x^p \int_X H_{x,\xi}D_\xi^p\phi(\xi)d\xi$, defined by the distribution $D_x^pD_\xi^pH_{x,\xi}$.

More generally, if D is a differential operator with C^∞ coefficients, of the form

$$DT = \sum_{|p| \leqq m} a_p(x)D^pT_x, \tag{8.4b}$$

its conjugate, its transpose and its adjoint are given by

$$\begin{aligned} \bar{D}T &= \sum_{|p| \leqq m} \overline{a_p}D^pT \\ {}^tDT &= \sum_{|p| \leqq m} (-1)^{|p|}D^p(a_pT) \\ D^*T &= \sum_{|p| \leqq m} (-1)^{|p|}D^p(\bar{a}_pT). \end{aligned} \tag{8.4c}$$

Then the image $D\mathscr{H}$ of $\mathscr{H}$ under D has its kernel defined by $\phi \mapsto D(H \cdot D^*\phi)$ so by the distribution $D_x\bar{D}_\xi H_{x,\xi}$.

If we apply Corollary 2 below, we have $D\mathscr{H} = 0$ if and only if $D_xH_{x,\xi} = 0$ or $\bar{D}_\xi H_{x,\xi} = 0$.

(43) See Schwartz [1], Chapter VII.

Corollary 1. *Let $\mathscr{M}$ be a Hilbert subspace of F, with kernel M. For $u(\mathscr{H}) \subset \mathscr{M}$ (resp. $u(\mathscr{H}) \supset \mathscr{M}$) to hold, it is necessary and sufficient that a constant $c \geqq 0$ exists such that*

$$uHu^* \leqq cM \qquad (\textit{resp. } M \leqq cuHu^*). \tag{8.5}$$

It suffices to apply Proposition 15.

Corollary 2. *We have $u(\mathscr{H}) = \{0\}$ (or $\mathscr{H} \subset u^{-1}(\{0\})$) if and only if $uH = 0$ (or $Hu^* = 0$).*

Indeed, $u(\mathscr{H}) = 0$ is equivalent to $\mathscr{K} = \{0\}$, where $\mathscr{K}$ is the orthogonal complement of $\mathscr{N}$ in $\mathscr{H}$. As $Hu^*(\bar{F}')$ is dense in $\mathscr{K}$ this is equivalent to $Hu^* = 0$, or $uH = 0$ by passing to adjoints $H^* = H$.

Corollary 3. *Let G be a dense vector subspace of E, equipped with a locally convex, quasi-complete topology such that the inclusion u of G in E is weakly continuous. Then E' is identified with a weakly dense subspace of G', with a weakly continuous inclusion u^*. For a Hilbert subspace $\mathscr{H}$ of E to be a Hilbert subspace of G, it is necessary and sufficient that its kernel H relative to E extends to a weakly continuous linear map $\hat{H} \geqq 0$ from G' into G. It is not necessary to assume that $\hat{H} \geqq 0$ if every point of G' is weakly adherent to a G'-bounded subset of E' (which always holds if G is a reflexive Banach space).*

Proof. Since u is injective, $u^*(\bar{E}')$ is weakly dense in G', and, since $u(G)$ is dense in E, u^* is injective; so u^* is a weakly continuous injection allowing to identify $\bar{E}'$ with a weakly dense subspace of $\bar{G}'$. It is then equivalent to saying that H extends to a non-negative kernel $\hat{H} : \bar{G}' \to G$, that is to say, that $H = uHu^* : \bar{E}' \xrightarrow{u^*} \bar{G}' \xrightarrow{H} G \xrightarrow{u} E$, or to saying that $\mathscr{H} = u(\mathscr{L}) = \mathscr{L}$, $\mathscr{L}$ being the Hilbert subspace of G with kernel H.

Let us now suppose that H extends to a kernel $\hat{H} : \bar{G}' \to G$, but that we do not know that $\hat{H} \geqq 0$. On the other hand, we suppose that every point g' of G' is weakly adherent to a G'-bounded subset $A_{g'}$ in E'; we have to show that $\mathscr{H}$ is again a Hilbert subspace of G. Let $\mathscr{H}_0 = H(\bar{E}') \subset G$. Let us show that the inclusion of $\mathscr{H}_0$ in G is continuous. Let $B_0 = \{H\bar{e}'; e' \in E', \langle H\bar{e}', e'\rangle \leqq 1\}$ be the unit ball of $\mathscr{H}_0$; we have to show that B_0 is weakly bounded in G, in other words, that, for every $g' \in G'$:

$$\sup_{e' \in E', \langle H\bar{e}', e'\rangle \leqq 1} \left|\langle H\bar{e}', g'\rangle\right| < +\infty. \tag{8.6}$$

As g' is weakly adherent to $A_{g'}$, and $H\bar{e}' \in G$, we have

$$|\langle H\bar{e}', g'\rangle| \leqq \sup_{f \in A_{g'}} \left|\langle H\bar{e}', f'\rangle\right| \leqq \langle H\bar{e}', e'\rangle^{1/2} \sup_{f' \in A_{g'}} \left\langle H\bar{f}', f'\right\rangle^{1/2} \tag{8.7}$$

such that in the end, it suffices to show that, for every $g' \in G'$:

$$\sup_{f' \in A_{g'}} \langle H\bar{f}', f'\rangle < +\infty; \tag{8.8}$$

which is evident since f' ranges over $A_{g'}$, bounded in G', and since $H\bar{f}'$ ranges over $\hat{H}(\bar{A}_{g'})$, bounded in G. Then the inclusion of $\mathscr{H}_0$ in E factorises into continuous inclusions $\mathscr{H}_0 \to G \to E$; by extending to quasi-completions, $\hat{\mathscr{H}_0} \to E$ factorises into $\hat{\mathscr{H}_0} \to G \to E$, and $\mathscr{H}$, the image of $\hat{\mathscr{H}_0}$ in E, is a Hilbert subspace of G. □

The condition pertaining to G' and E' always holds if G is a reflexive Banach space, or more generally the dual of a reflexive Fréchet space. This is because G' with the strong topology is a Fréchet space with dual G, E' is strongly dense in G' and every point of G' is the limit in G' of a sequence of elements of E'.

Remark 1. In fact, if we know that $\hat{H} \geqq 0$, there is a trivial direct proof for Corollary 3. If $\mathscr{H} \subset E$ is a Hilbert subspace of G, it has a kernel $\hat{H} : \bar{G}' \to G$; the characteristic relationship (4.2) shows that its kernel $H : \bar{E}' \to E$ is the restriction of $\hat{H}$. Conversely, if $\mathscr{H}$ is a Hilbert subspace of E, and if its kernel $H : \bar{E}' \to E$ extends to a non-negative kernel $\hat{H} : \bar{G}' \to G$, then $\hat{H}$ is the kernel relative to G of a Hilbert subspace $\mathscr{L}$ of G; the kernel of $\mathscr{L}$ in E is the restriction of $\hat{H}$, by the preceding assertion, so it is H, and therefore $\mathscr{L} = \mathscr{H}$, which is thus a Hilbert subspace of G.

Remark 2. We distinguished H and its extension $\hat{H}$. *In practice, we generally also denote the extension to* $\bar{G}' \to G$ *by* H.

Example. Let $\mathscr{H}$ be a Hilbert subspace of $\mathscr{D}'$, the space of distributions on $\mathbb{R}^n$. Its kernel $H_{x,\xi}$ is a distribution on $\mathbb{R}^n \times \mathbb{R}^n$ (Example 3, page 24). For $\mathscr{H}$ to be a Hilbert subspace of one of the spaces $E = \mathscr{D}, \mathscr{E}', \mathscr{E}, \mathscr{S}', \mathscr{S}, \mathscr{O}'_M, \mathscr{O}_M, \mathscr{O}'_C, \mathscr{O}_C$, it is necessary and sufficient that $H_{x,\xi}$ belongs to $E_x \hat{\otimes}_\varepsilon E_\xi$ and is non-negative in this space; but this non-negativity is automatic, by the criterion of Corollary 3, because every element T of E' is adherent to an E'-bounded subset of $\mathscr{D}$. [Let α_v, $v = 1, 2, ...$ be a sequence of functions of $\mathscr{D}$ having the following property: for every p, $\mathscr{D}^p(\alpha_v - 1)$ converges to 0 uniformly on every compact subset of $\mathbb{R}^n$ as $v \to +\infty$, while staying bounded in $\mathbb{R}^n$. Let ρ_μ be a sequence of non-negative functions in $\mathscr{D}$, with supports converging to the origin, with $\int \rho_\mu = 1$. Then, in E', $T = \lim_{\mu\to\infty}(\lim_{v\to\infty} \alpha_v(\rho_\mu * T))$, and the set of the $\alpha_v(\rho_\mu * T) \in \mathscr{D}$ is bounded in E'].

Corollary 4. *The map* $\mathscr{H} \to u(\mathscr{H})$ *of* Hilb(E) *into* Hilb(F)*, defined by a continuous linear or antilinear map* u *from* E *into* F*, is a homomorphism with respect to multiplication by non-negative real numbers, addition and the order relation:*

$$\begin{aligned} &u(\lambda\mathscr{H}) = \lambda u(\mathscr{H}); \\ &u(\mathscr{H}_1 + \mathscr{H}_2) = u(\mathscr{H}_1) + u(\mathscr{H}_2); \\ &\mathscr{H}_1 \leqq \mathscr{H}_2 \quad \Longrightarrow \quad u(\mathscr{H}_1) \leqq u(\mathscr{H}_2). \end{aligned} \tag{8.9}$$

Proof. It is easy to see directly but even easier by noticing that $H \to uHu^*$ is a homomorphism with respect to the corresponding structures of $\mathscr{L}^+(E)$ and $\mathscr{L}^+(F)$. □

Corollary 5. *Let* E *be a Hilbert space,* $\mathscr{H}$ *a Hilbert subspace of* E *and* H *its kernel* $E \to \mathscr{H}$ *(Example 4, page 25). Then* $\mathscr{H}$ *is the image* $\sqrt{H}E$ *of* E *under* $\sqrt{H}$.

Proof. $\sqrt{H}$ is its own adjoint. The kernel of E in itself being the identity I, that of $\sqrt{H}E$ is, according to Proposition 21, $\sqrt{H}I\sqrt{H} = H$; we thus have $\sqrt{H}E = \mathscr{H}$. □

This gives, in the case where E is Hilbert, a direct construction of $\mathscr{H}$ from H. $\mathscr{H}$ is the set of the $\sqrt{H}\xi$, $\xi \in E$, and we have

$$\|\eta\|_{\mathscr{H}} = \inf_{\sqrt{H}\xi=\eta} \|\xi\| . \tag{8.9b}$$

Transport of structure.

Let u be a weak isomorphism (resp. anti-isomorphism) from E onto F. It automatically defines, by transport (resp. anti-transport) of structure:

a) a weak isomorphism (resp. anti-isomorphism) from $\bar{E}$ onto $\bar{F}$, which is nothing but $\bar{u}$, but which we will also denote by u; so that $u\bar{e} = \overline{ue}$.

b) a weak isomorphism (resp. anti-isomorphism) from $\bar{E}'$ onto $\bar{F}'$, which is nothing but $(u^*)^{-1}$, but which we will also denote by u:

$$\langle u\bar{e}', u\bar{e}\rangle = \langle \bar{e}', \bar{e}'\rangle \qquad (\text{resp. } \overline{\langle \bar{e}'\bar{e}\rangle}); \tag{8.10}$$

c) an isomorphism (resp. anti-isomorphism) from $\mathscr{L}(\bar{E}'; E)$ onto $\mathscr{L}(\bar{F}'; F)$ equipped with the topology of weak pointwise convergence, which is nothing but $H \mapsto uHu^*$, but which we will denote again by $u : H \mapsto u(H)$

$$u(H)(u(\bar{e}')) = u(H\bar{e}'). \tag{8.11}$$

Finally, if $\mathscr{H}$ is a Hilbert subspace of E, $u(\mathscr{H})$ is a Hilbert subspace of F defined as the image of $\mathscr{H}$ under u with the transport (resp. anti-transport) of the scalar product:

$$(uk \mid uk)_{u(\mathscr{H})} = (h \mid k)_{\mathscr{H}} \quad (\text{resp. } \overline{(h \mid k)_{\mathscr{H}}}) \tag{8.12}$$

$$\|uh\|_{u(\mathscr{H})} = \|h\|_{\mathscr{H}}.$$

Then, as a result of the intrinsic character of the definition of the kernel H of a Hilbert subspace $\mathscr{H}$ of E, the transport of structure shows that the kernel of $u(\mathscr{H})$ in $u(F)$ is $u(H)$; it is a particular case of Proposition 21, since $u(H) = uHu^*$.

Let us suppose, for example, that $F = E$, with u being the conjugation. Then $\bar{F} = E$, with the reciprocal map of the conjugation. We have: $ue = \bar{e}$, $u\bar{e} = e$, $ue' = \bar{e}'$ and $u\bar{e}' = e'$; finally, (8.11) gives $u(H) \cdot e' = \overline{H\bar{e}'} = \bar{H}e'$ or $u(H) = \bar{H}$. If then $\mathscr{H}$ is a Hilbert subspace of E, with kernel H, its conjugate $\bar{\mathscr{H}}$ (with the transport of the norm) has kernel $\bar{H}$.

Invariance under an automorphism. If, in particular, $F = E$, u becomes a weak automorphism (resp. anti-automorphism) of E. We will say that $\mathscr{H}$ is invariant under u if $u(\mathscr{H}) = \mathscr{H}$; this means that u is a unitary (resp. anti-unitary) operator of $\mathscr{H}$; for it to be so, it is necessary and sufficient that the kernel H is invariant under u, that is to say, $u(H) = H$, or $uHu^* = H$, or $uHu^{-1} = H$, or $uH = Hu$ (as a map from $\bar{E}'$ into E). If we suppose that u ranges over a group G of weak automorphisms of E, and if $\mathscr{H}$ is invariant under G, we have a unitary representation of G in $\mathscr{H}$; the invariant kernels under G form a convex sub-cone of $\mathscr{L}^+(\bar{E}', E)$, closed with respect to the topology of weak pointwise convergence. These are some properties that we have systematically applied in the study of elementary relativistic particles in quantum mechanics (related to some irreducible unitary representations of the inhomogeneous Lorentz group). In the same vein, let us suppose that E is equipped with an anti-involution. Then a Hilbert subspace $\mathscr{H}$ of E is invariant under this anti-involution ($\bar{\mathscr{H}} = \mathscr{H}$, with the transport of norms), if and only if H is invariant, $\bar{H} = H$.

Multiplication of measures spaces.

Proposition 22. *Let μ be a non-negative Radon measure on a locally compact space X, and let p be a locally square-μ-integrable complex function. Then the image $p\Lambda^2(X,\mu)$ of $\Lambda^2(X,\mu)$ under multiplication by p is $\Lambda^2(X,|p|^2\mu)$, and its kernel in $\mathscr{D}_c'^0(X)$ (the space of measures on X) is $\phi \mapsto \phi|p|^2\mu$. If p is everywhere $\neq 0$, multiplication by p is an isometry from $\Lambda^2(X,\mu)$ onto $\Lambda^2(X,|p|^2\mu)$; moreover, this latter is the space of measures of the form $h\mu$, $h/p \in L^2(X,\mu)$, with*

$$\|h\mu\|_{\Lambda^2(X,|p|^2\mu)} = \left\|\frac{h}{p}\right\|_{L^2(X,\mu)}. \tag{8.13}$$

Proof. We cannot use Example 2 on page 43, because p does not operate by multiplication on $\mathscr{D}'^0(X)$, as it is not necessarily continuous.

1°) Let σ be a measure in $\Lambda^2(X,\mu)$; $\sigma = f\mu$, $f \in L^2(X,\mu)$ and

$$\|\sigma\|_{\Lambda^2(X,\mu)} = \|f\|_{L^2(X,\mu)}. \tag{8.14}$$

As p and f are locally square-μ-integrable, pf is locally μ-integrable, so $\tau = p\sigma = pf\mu$ is a measure. Let us denote by g a μ-measurable function such that $pf = g|p|^2$; g is well-defined on a set $Y = \{x \in X, p(x) \neq 0\}$, and is set to be any arbitrary measurable function on the complement Y^c. We see that g is square-$|p|^2\mu$-integrable:

$$\begin{aligned} \|f\|^2{}_{L^2(X,\mu)} &= \int_X |f|^2\,d\mu \geqq \int_Y |f|^2\,d\mu \\ &= \int_Y |g|^2\,|p|^2\,d\mu = \int_X |g|^2\,|p|^2\,d\mu = \|g\|^2_{L^2(X,|p|^2\mu)}. \end{aligned} \tag{8.15}$$

As $\tau = g(|g|^2\mu)$, we have $p\sigma = t \in \Lambda^2(X,|p|^2\mu)$. (8.15) then gives:

$$\|p\sigma\|_{\Lambda^2(X,|p|^2\mu)} \leqq \|\sigma\|_{\Lambda^2(X\mu)}. \tag{8.16}$$

2°) Conversely, let $\tau \in \Lambda^2(X,|p|^2\mu)$; we have

$$\tau = g|p|^2\mu,$$

and

$$\|\tau\|_{\Lambda^2(X,|p|^2\mu)} = \|g\|_{L^2(X,|p|^2\mu)}. \tag{8.17}$$

We can find some μ-measurable functions f such that $pf = g|p|^2$; they are well-defined on Y, and (measurably) arbitrary on the complement Y^c. Then, for each of these f, $\int_Y |f|^2 d\mu < +\infty$; if we choose f on Y^c such that $\int_X |f|^2 d\mu < +\infty$, we will have $\sigma = f\mu \in \Lambda^2(X,\mu)$, and $\tau = p\sigma$. So we have $\tau \in p\Lambda^2(X,\mu)$. Moreover, the inequality (8.15) is an equality if we choose $f = 0$ on Y^c. (8.15) then becomes

$$\|\tau\|_{\Lambda^2(X,|p|^2\mu)} = \inf_{p\sigma=\tau} \|\sigma\|_{\Lambda^2(X,\mu)}. \tag{8.18}$$

1°) and 2°) show that $p\Lambda^2(X,\mu) = \Lambda^2(X,|p|^2\mu)$.

3°) We can reprove it using Proposition 21. The map $\sigma \mapsto p\sigma$ is continuous and linear from $\Lambda^2(X,\mu)$ into $\mathscr{D}_c'^0(X)$. To see it, it suffices to show that it is weakly continuous; but, if $f\mu$ converges weakly to 0 in $\Lambda^2(X,\mu)$, $pf\mu$ converges weakly to 0 in $\mathscr{D}'^0(X)$, because of

$$(8.19)\qquad \langle pf\mu, \phi\rangle_{\mathscr{D}'^0,\mathscr{D}} = \langle f\mu, p\phi\mu\rangle_{\Lambda^2(\mu),(\Lambda^2(\mu))'} = \int_X fp\phi d\mu.$$

(Equation (1.2d)).

This proves at the same time that the transpose of the map $f\mu \mapsto pf\mu$ from $\Lambda^2(X,\mu)$ into $\mathscr{D}_c'^0(X)$ is the map $\phi \mapsto p\phi\mu$ from $\mathscr{D}^0(X)$ into $\Lambda^2(X,\mu)$, identified with its dual. Then the adjoint of $f(\mu) \mapsto pf\mu$ is $\bar{\phi} \mapsto \bar{p}\bar{\phi}\mu$ or $\phi \mapsto \bar{p}\phi\mu$.

Let us then apply Proposition 21. The kernel of $\Lambda^2(X,\mu)$ relative to itself is the identity (by identifying $(\Lambda^2)'$ with Λ^2). So the kernel of the image $p\Lambda^2(X,\mu)$ in $\mathscr{D}_c'^0(X)$ is $\phi \mapsto p\bar{p}\phi\mu = |p|^2\phi\mu$. It is also the kernel of $\Lambda^2(X,|p|^2\mu)$ (Example 1c on page 23), which indeed shows that these two spaces are identical.

4°) The results in the case where p is everywhere $\neq 0$ are obvious, because (8.15) becomes an equality. For $\tau \in p\Lambda^2(X,\mu)$, $\tau = p\sigma = pf\mu$, we set $\tau = h\mu$, $h = pf$, and we have $h/p = f \in L^2(X,\mu)$ and vice versa; moreover,

$$(8.20)\qquad \|\tau\|_{p\Lambda^2(X,\mu)} = \|\sigma\|_{\Lambda^2(X\mu)} = \|f\|_{L^2(X,\mu)} = \left\|h/p\right\|_{L^2(X,\mu)}$$

which is (8.16).

□

Remark. If X is an open subset of $\mathbb{R}^n$, and if $d\mu = dx$, we generally identify f with fdx, and $L^2(X,dx)$ with $\Lambda^2(X,dx)$. Then pL^2 is the space $\Lambda^2(X,|p|^2dx)$, and its kernel in $\mathscr{D}'(X)$ is $\phi \mapsto |pr|^2\phi$ (as a linear map from $\mathscr{D}$ into $\mathscr{D}'$). If p is everywhere $\neq 0$, pL^2 is the space of the functions h such that $\frac{h}{p} \in L^2$, with

$$(8.21)\qquad \|h\|_{pL^2} = \left\|\frac{h}{p}\right\|_{L^2}.$$

Expressing inclusion relations by upper bounds.

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert subspaces of E. The inclusion relation $\mathscr{H} \leqq \mathscr{K}$ (which means, as we have seen on page 15, that $\mathscr{H} \subset \mathscr{K}$ and that the unit ball of $\mathscr{H}$ is contained in the unit ball of $\mathscr{K}$) is expressed, according to Proposition 13, by the inequality between kernels $H \leqq K$, or equivalently: for every $e' \in E'$, $\langle H\bar{e}', e'\rangle \leqq \langle K\bar{e}', e'\rangle$. Now let u and v weakly continuous maps from E into E; u^* and v^* are then weakly continuous from $\bar{E}'$ into $\bar{E}'$.

The inclusion relation $u(\mathscr{H}) \leqq v(\mathscr{K})$, which expresses that $u(\mathscr{H}) \subset v(\mathscr{K})$ and that the image under u of the unit ball of $\mathscr{H}$ is contained in the image under

v of the unit ball of $\mathscr{K}$, will be expressed by the inequality between kernels:

$$(8.22) \qquad \begin{cases} uHu^* \leqq vKv^*, & \text{or equivalently, for every } e' \in E', \\ \langle uHu^*\bar{e}', e'\rangle \leqq \langle vKv^*e', e'\rangle, & \text{or} \\ (Hu^*\bar{e}' \mid u^*\bar{e}')_{E,\bar{E}'} \leqq (Kv^*\bar{e}' \mid v^*\bar{e}')_{E,\bar{E}'} & \text{or, by (4.3):} \\ \left\|Hu^*\bar{e}'\right\|_{\mathscr{H}} \leqq \left\|Kv^*\bar{e}'\right\|_{\mathscr{K}}. \end{cases}$$

It is convenient to let u and v step in, instead of u^* and v^*; so we use (4.7), which gives:

$$(8.23) \qquad \left\|Hu^*\bar{e}'\right\|_{\mathscr{H}} = \sup_{f' \in E'} \frac{\left|\langle Hu^*\bar{e}', f'\rangle\right|}{\langle H\bar{f}', f'\rangle^{1/2}} = \sup_{f' \in E'} \frac{\left|\langle uH\bar{f}', e'\rangle\right|}{\langle H\bar{f}', f'\rangle^{1/2}}$$

so that we will have $u(\mathscr{H}) \leqq v(\mathscr{K})$, if and only if, for every $e \in E'$:

$$(8.24) \qquad \sup_{f' \in E'} \frac{\left|\langle uH\bar{f}', e'\rangle\right|}{\langle H\bar{f}', f'\rangle^{1/2}} \leqq \sup_{f' \in E'} \frac{\left|\langle vK\bar{f}', e'\rangle\right|}{\langle K\bar{e}', e'\rangle^{1/2}}.$$

We will restrict ourselves to $v =$ identity; then it is in our interests to keep the right-hand side under the form $\|K\bar{e}'\|_{\mathscr{K}} = \langle K\bar{e}', e'\rangle^{1/2}$.

But we will successively consider $\leqq$ and $\geqq$. So:

Proposition 22b. *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert subspaces of E, and let u be a continuous linear map from E into E.*

1°) *The necessary and sufficient condition for $u(\mathscr{H}) \leqq \mathscr{K}$ is that, for any $e', f' \in E'$, we have:*

$$(8.25) \qquad \left|\langle uH\bar{f}', e'\rangle_{E,E'}\right| \leqq \langle H\bar{f}', f'\rangle^{1/2}\langle K\bar{e}', e'\rangle^{1/2}.$$

This condition is verified if, for every $e' \in E'$, we have:

$$(8.26) \qquad \left|\langle uH\bar{e}', e'\rangle_{E,E'}\right| \leqq \frac{1}{2}\langle H\bar{e}', e'\rangle^{1/2}\langle K\bar{e}', e'\rangle^{1/2}.$$

2°) *The necessary and sufficient condition for $u(\mathscr{H}) \geqq \mathscr{K}$ is that, for every $e' \in E'$, we have*

$$(8.27) \qquad \sup_{f' \in E'} \frac{\left|\langle uH\bar{f}', e'\rangle_{E,E'}\right|}{\langle H\bar{f}', f'\rangle^{1/2}} \geqq \langle K\bar{e}', e'\rangle^{1/2}.$$

This condition is verified if, for any $e' \in E'$, we have:

$$(8.28) \qquad \left|\langle uH\bar{e}', e'\rangle_{E,E'}\right| \geqq \langle H\bar{e}', e'\rangle^{1/2}\langle K\bar{e}', e'\rangle^{1/2}.$$

Everything results from the preceding arguments; the implication (8.28) $\Longrightarrow$ (8.27) is trivial, while (8.26) $\Longrightarrow$ (8.25) was seen on page 30.

The relationship (8.28), in general replaced by the even stronger relationship:

$$(8.29) \qquad \mathrm{Re}\langle uH\bar{e}', e'\rangle_{E,E'} \geqq \langle H\bar{e}', e'\rangle^{1/2}\langle K\bar{e}', e'\rangle^{1/2},$$

is of a type known under the name of coercivity relation.

The functors Hilb and $\mathscr{L}^+$ and their isomorphism

A *convex cone* refers to a set Γ, equipped with the following structures:

1°) a law of multiplication by non-negative scalars: $(\lambda, \xi) \mapsto \lambda\xi$, a map from $\mathbb{R}_+ \times \Gamma$ into Γ, with the properties: $1\xi = \xi$, $(\lambda\mu)\xi = \lambda(\mu\xi)$.

2°) an addition law, $(\xi, \eta) \mapsto \xi + \eta$, a map from $\Gamma \times \Gamma$ into Γ, which is associative, commutative and equipped with a neutral element 0.

We assume moreover that these two laws are connected by the following properties: $(\lambda + \mu)\xi = \lambda\xi + \mu\xi$, $\lambda(\xi + \eta) = \lambda\xi + \lambda\eta$ (distributivity of multiplication with respect to addition) and $0\xi = \lambda 0 = 0$.

We say that the cone is salient, or strictly convex, if $\xi + \eta = 0$ implies $\xi = \eta = 0$. We say that it is regular if $\xi + \zeta = \eta + \zeta$ implies $\xi = \eta$. A convex salient cone of a vector space over $\mathbb{R}$ (see page 16) satisfies all of these properties; and it is essentially the only case, because we can associate, uniquely up to isomorphism, a vector space G to any regular salient convex cone Γ, such that Γ is a salient convex cone of G and generates G (an element of G is an equivalence class of $\Gamma \times \Gamma$, with $(\xi, \xi') \sim (\eta, \eta')$ if $\xi + \eta' = \xi' + \eta$; the class of (ξ, ξ') becomes the difference $\xi - \xi'$ in the vector space G).

On a salient convex cone Γ, we canonically define an order relation by $\xi \leqq \eta$ if there exists $\zeta \in \Gamma$ (necessarily unique) such that $\eta = \xi + \zeta$. Then, $\xi \leqq \xi'$ and $\eta \leqq \eta'$ imply $\xi + \eta \leqq \xi' + \eta'$, $\xi \leqq \eta$ is equivalent to $\xi + \zeta \leqq \eta + \zeta$, $\xi \leqq \eta$ implies $\lambda\xi \leqq \lambda\eta$ and, for $\xi \neq 0$, $\lambda\xi \leqq \xi$ is equivalent to $\lambda \leqq 1$.

If G is a vector space generated by Γ, G is an ordered vector space, where Γ is the cone of non-negative elements.

Let $\mathscr{E}$ be the category of locally convex, quasi-complete Hausdorff topological vector spaces on $\mathbb{C}$, the morphisms being the continuous linear maps. Let $\mathscr{G}$ be the category of regular salient convex cones, with the morphisms being the maps that preserve multiplication by non-negative scalars and addition, and so the order relation as well.

Then Hilb : $E \mapsto \text{Hilb}(E)$ is a covariant functor from the category $\mathscr{E}$ into the category $\mathscr{G}$, if we associate the morphism $\mathscr{H} \mapsto u(\mathscr{H}) : \text{Hilb}(E) \to \text{Hilb}(F)$ to a morphism $u : E \to F$.

Furthermore, $\mathscr{L}^+ : E \to \mathscr{L}^+(E) = \mathscr{L}^+(\bar{E}'; E)$ is another covariant functor from the category $\mathscr{E}$ into the category $\mathscr{G}$, if we associate the morphism $H \mapsto uHu^* : \mathscr{L}^+(E) \to \mathscr{L}^+(F)$ to a morphism $u : E \to F$.

Finally, the different Propositions of the preceding subsections show that these two functors Hilb and $\mathscr{L}^+$ are isomorphic, the isomorphism being the canonical map $\text{Hilb}(E) \to \mathscr{L}^+(E)$, for $E \in \mathscr{E}$.

We can naturally introduce the vector space generated by Hilb(E) (set of classes of formal differences of Hilbert subspaces of E); it is isomorphic to vector subspaces (over $\mathbb{R}$) of $\mathscr{L}(\bar{E}'; E)$ generated by positive kernels. We will study it later (§12).

§9. Spaces of functions on a set X. Reproducing kernel of Aronszajn-Bergman

Let X be a set, and $E = \mathbb{C}^X$ the space of complex functions on X, equipped with the topology of pointwise convergence. The dual $E' = (\mathbb{C}^X)'$ is the space of measures with finite support on X; such a measure is of the form $\mu = \sum_{x \in X} c_x \delta_{(x)}$ where $\delta_{(x)}$ is the Dirac measure of the point x, and where $c_x \in \mathbb{C}$, *except a finite number of them, are zero*; we have, for $f \in \mathbb{C}^X$:

$$\langle \mu, f \rangle = \sum c_x f(x). \tag{9.1}$$

Let γ be a linear form on $(\mathbb{C}^X)'$; we can let $g(x) = \langle \gamma, \delta_{(x)} \rangle$, and then g is a function on X, so an element of $\mathbb{C}^X$, and we have $\langle \gamma, \mu \rangle = \sum_{x \in X} c_x g(x) = \langle \mu, g \rangle$ if $\mu = \sum_{x \in X} c_x \delta_{(x)}$. Thus every linear form on E' is weakly continuous and consequently so is every linear map from E' into a topological vector space.

Naturally E and E' are equipped with contragredient anti-involutions, defined by the complex conjugation $f \mapsto \bar{f}$ and $\mu \mapsto \bar{\mu}$.

Proposition 23. *Let H be a kernel relative to $E = \mathbb{C}^X$; the associated reproducing kernel A is the function on $X \times X$ defined by*

$$A(x, \xi) = \langle H\delta_{(\xi)}; \delta_{(x)} \rangle, \qquad x \in X, \xi \in X. \tag{9.2}$$

Then $H \mapsto A$ is an isomorphism of the space $\mathscr{L}(\bar{E}'; E)$ of the kernels relative to E, equipped with the topology of weak pointwise convergence, onto the space $\mathbb{C}^{X \times X}$ of complex functions on $X \times X$, equipped with the topology of pointwise convergence. H is defined from A by

$$\begin{cases} \langle H\bar{v}, \mu \rangle = \sum_{(x,\xi) \in X \times X} c_x \bar{d}_\xi A(x, \xi) \\ \textit{for } \mu = \sum_{x \in X} c_x \delta_{(x)}, \quad v = \sum_{\xi \in X} d_\xi \delta_{(\xi)}; \end{cases} \tag{9.3}$$

Hv is the function $x \mapsto \sum_{\xi \in X} d_\xi A(x, \xi)$; in particular, $H\delta_{(\xi)}$ is the function $A(, \xi) : x \mapsto A(x, \underline{\xi})$; and the reproducing kernel associated to H^* is the symmetric conjugate $\overline{{}^s A}$ defined by:

$$\overline{{}^s A}(x, \xi) = \overline{A(\xi, x)}; \tag{9.4}$$

H is Hermitian (i.e. $H^* = H$) if and only if $\overline{{}^s A} = A$ or

$$A(x, \xi) = \overline{A(\xi, x)}. \tag{9.5}$$

$H \geqq 0$ if and only if A is "of positive type":

$$\forall \mu = \sum_{x \in X} c_x \delta_{(x)}, \qquad \sum_{x,\xi} c_x \bar{c}_\xi A(x, \xi) \geqq 0. \tag{9.6}$$

Proof. This Proposition is evident. (9.2) defines A from H; and what follows defines H from A, and H is certainly weakly continuous following what was said before the statement of the Proposition. (9.4) is obvious from (9.2), and hence so is (9.5). (9.6) precisely translates the non-negativity of H: $\forall \mu \in (\mathbb{C}^X)'$, $\langle H\bar{\mu}, \mu \rangle \geqq 0$. □

Corollary. *There exists an isomorphism between the set $\mathscr{H}(\mathbb{C}^X)$ of the Hilbert subspaces of $\mathbb{C}_X$ (equipped with the structures defined in §2) and the salient convex cone of $\mathbb{C}^{X\times X}$ consisting of functions of positive type. The function A on $X \times X$ associated to the Hilbert subspace $\mathscr{H}$ of $\mathbb{C}^X$ is characterised by:*

$$\forall h \in \mathscr{H}, \forall x \in X, \qquad (h \mid A(,x))_{\mathscr{H}} = h(x). \tag{9.10}$$

In particular:

$$\begin{cases} \forall x \in X, \forall \xi \in X, (A(,\xi) \mid A(,x))_{\mathscr{H}} = A(x,\xi) \\ \|A(,x)\|^2_{\mathscr{H}} = A(x,x); \qquad \textit{and} \end{cases} \tag{9.11}$$

$$\left|h(x)\right| \leqq \|h\| \, (A(x,x))^{1/2}. \tag{9.11b}$$

Proof. (9.10) is (4.2) applied to $e' = \delta_{(x)}$; and conversely, from (9.10), we deduce (4.2) by taking the combination $e' = \mu = \sum\limits_{x \in X} c_x \delta_{(x)}$. From (9.10) we deduce (9.11) by taking $h = A(,\xi)$. (9.11b) is the Cauchy-Schwarz inequality applied to (9.10) taking into account (9.11). □

It is Bergman who has first introduced the kernel A, notably in the theory of analytic and harmonic functions; Aronszajn studied it in all its generality for all Hilbert spaces of functions on a set X. He called A the reproducing kernel, because of (9.10). He proved the above Corollary, that is to say, the isomorphism theorem of §6 for the particular case $E = \mathbb{C}^X$. This case $E = \mathbb{C}^X$ is clearly much more specific than ours. But as every locally convex vector space E is a subspace of the space $\mathbb{C}^{E'}$ of complex functions on E' and has a finer topology than that of pointwise convergence on E' (which is the weak topology of E), every Hilbert subspace $\mathscr{H}$ of E is a fortiori a Hilbert subspace of $\mathbb{C}^{E'}$! We could thus, in this manner, deduce many of the preceding Propositions (notably the isomorphism theorem) from that of Aronszajn, arriving at a complement allowing to pass from $\mathbb{C}^{E'}$ to the subspace E (and inevitably having to use the quasi-complete character of E). This complement is, essentially, Proposition 0: every Hilbert subspace of $\mathbb{C}^{E'}$ having a dense subspace in E is a Hilbert subspace of E. We could thus have shortened the exposition by assuming the theory of Bergman-Aronszajn to be known and by applying it as it is. The reason we have not done that is because the results are *just as easy, often even easier* to show *directly* for any E, than for the particular case $E = \mathbb{C}^X$. The non-negativity $\langle He', e'\rangle \geqq 0$ is simpler than $\sum c_x \bar{c}_\xi A(x,\xi) \geqq 0$.

Proposition 24. *Let X be a locally compact topological space. Let $\mathscr{E}^0(X) \subset \mathbb{C}^X$ be the space of continuous complex functions on X, equipped with the topology of uniform convergence on all compact subsets; its dual $\mathscr{E}'^0(X)$ is the space of Radon measures with compact support on X. Let $\mathscr{H}$ be a Hilbert subspace of $\mathbb{C}^X$, and A its reproducing kernel of Aronszajn. For $\mathscr{H}$ to be a Hilbert subspace of $\mathscr{E}^0(X)$, it is necessary and sufficient that A is separately continuous on $X \times X$, and locally bounded. In this case, its kernel H relative to $\mathscr{E}'^0(X)$ is defined as follows: if $\nu \in \mathscr{E}'^0(X)$, $H\nu$ is the continuous function*

$$(H\nu)(x) = \int_X A(x,\xi) d\nu(\xi); \tag{9.12}$$

moreover:

$$\langle H\bar{\nu}, \mu\rangle = \int_X d\mu(x) \int_X A(x,\xi) d\bar{\nu}(\xi) \tag{9.13}$$
$$= \int_X d\bar{\nu}(\xi) \int_X A(x,\xi) d\mu(x), \qquad \textit{and}$$
$$\langle H\bar{\mu}, \mu\rangle = \int_X d\mu(x) \int_X A(x,\xi) \overline{d\mu(\xi)} \geqq 0. \tag{9.14}$$

For the inclusion of $\mathscr{H}$ in $\mathscr{E}^0(X)$ to be compact, it is necessary for A to be a continuous function on $X \times X$, and it is sufficient for it to be separately continuous, and continuous on the diagonal of $X \times X$.

Proof. 1°) Let us suppose that $\mathscr{H}$ is a Hilbert subspace of $\mathscr{E}^0(X)$; let H be its kernel relative to $\mathscr{E}^0(X)$, and A its reproducing kernel, as a Hilbert subspace of $\mathbb{C}^X$. The map $x \mapsto \delta_{(x)}$ is continuous from X into $\mathscr{E}'^0(X)$ with respect to the weak topology; since then the sesquilinear form $\tilde{H}$ defined by H (Equation (3.5), Proposition 5) is separately weakly continuous, A defined by (9.2) is separately continuous on $X \times X$ (we can also say: $A(\cdot,\xi) \in \mathscr{H}$ so $A(\cdot,\xi) \in \mathscr{E}^0(X)$, so $x \mapsto A(x,\xi)$ is continuous; likewise $\xi \mapsto A(\xi,x) = A(x,\xi)$). If $(x,\xi) \in K \times K$, with K a compact subset of X, $\delta_{(x)}$ ranges over a weakly compact, and hence bounded, subset of $\mathscr{E}'^0(X)$, and $H\delta_{(\xi)}$ ranges over a weakly compact, and hence bounded, subset of $\mathscr{E}^0(X)$, so $A(x,\xi) = \langle H\delta_{(\xi)}, \delta_{(x)}\rangle$ is bounded on $K \times K$: A is locally bounded on $X \times X$.

2°) Conversely, let $\mathscr{H}$ be a Hilbert subspace of $\mathbb{C}^X$, and A its reproducing kernel of Aronszajn, and let us suppose that A is separately continuous and locally bounded. Let us suppose that $x \in X$ converges to $x_0 \in X$ while staying in a compact set; then $\|A(\cdot,x)\|^2_{\mathscr{H}} = A(x,x)$ stays bounded, and $(A(\cdot,\xi) \mid A(\cdot,x))_{\mathscr{H}} = A(x,\xi)$ converges to $A(x_0,\xi) = (A(\cdot,\xi) \mid A(\cdot,x_0))$; but the $A(\cdot,\xi)$, $\xi \in X$, form a total set in $\mathscr{H}$ (since the $H\mu$, $\mu \in (\mathbb{C}^X)'$, which are finite linear combinations of these, form the dense subspace $H((\mathbb{C}^X)') = \mathscr{H}_0$ of Proposition 7); so $A(\cdot,x)$ converges to $A(\cdot,x_0)$ weakly in $\mathscr{H}$(44). Then, for $h \in \mathscr{H}$, $(h \mid A(\cdot,x))_{\mathscr{H}} = h(x)$ will converge to $(h \mid A(\cdot,x_0)) = h(x_0)$, so each h is a continuous function on X. So $\mathscr{H} \in \mathscr{E}^0(X)$, and the closed graph theorem(45) shows that its inclusion is continuous, and hence that it is a Hilbert subspace of $\mathscr{E}^0(X)$.

3°) Let us suppose that the preceding properties are satisfied. We have, for ν in $\mathscr{E}'^0(X)$:

$$\nu = \int_X \delta_{(\xi)} d\nu(\xi), \tag{9.15}$$

because this simply expresses that, for every $f \in \mathscr{E}^0(X)$, we have:

$$\langle \nu, f\rangle = \int f(\xi) d\nu(\xi). \tag{9.16}$$

(44)In accordance with Ascoli's theorem (Bourbaki [1], Chapter III, §3, n°5, Proposition 5).
(45)Bourbaki [1], Chapter I, §3, n°3, Corollary 5 of Theorem 1.

As $\nu \to H\nu(x) = \langle H\nu, \delta_{(x)} \rangle$ is a continuous linear form on $\mathscr{E}'^0(X)$ with the weak topology, we deduce from (9.15) that:

$$(9.17) \qquad (H\nu)(x) = \int_X \langle H\delta_{(\xi)}, \delta_{(x)} \rangle d\nu(\xi) = \int_X A(x,\xi) d\nu(\xi),$$

which is (9.12); the last term is the integral of a continuous function with respect to a Radon measure with compact support; moreover, the obtained result, which is $(H\nu)(x)$, is necessarily a continuous function of $x \in X$.

Then we will have, by integrating with respect to μ the continuous function $H\bar{\nu}$

$$(9.18) \qquad \langle H\bar{\nu}, \mu \rangle = \int ((H\bar{\nu})(x)) d\mu(x) = \int_X d\mu(x) \int_X A(x,\xi) \overline{d\nu(\xi)},$$

which is the first equality of (9.13).

In addition, $H\delta_{(\xi)} = A(\cdot, \xi)$, so

$$(9.19) \qquad \langle H\delta_{(\xi)}, \mu \rangle = \int A(x,\xi) d\mu(x);$$

as $\nu \to \langle H\bar{\nu}, \mu \rangle$ is continuous and linear on $\mathscr{E}'^0(X)$ with the weak topology, we deduce from (9.15) that:

$$\begin{aligned} \langle H\bar{v}, \mu \rangle &= \int_X \langle H\delta_{(\xi)}, \mu \rangle \overline{d\nu(\xi)} \\ &= \int_X \overline{d\nu(\xi)} \int_X A(x,\xi) d\mu(x), \end{aligned}$$

which is the second equality of (9.13).

By setting $\nu = \mu$, we obtain (9.14), which is by no means evident a priori if we only suppose A to be separately continuous, locally bounded and of positive type. Naturally, we would see it directly by the application of the criterion of Corollary 3 of Proposition 21: every measure μ of $\mathscr{E}'^0(X)$ is a weak limit of finite combinations of point masses, in other words, it is an element of $(\mathbb{C}^X)'$, with supports contained in that of μ and with norm $\leqq \|\mu\|$, so forming a bounded subset of $\mathscr{E}'^0(X)$.

4°) According to Proposition 9c, $\mathscr{H}$ has a compact inclusion in $\mathscr{E}'^0(X)$ if and only if, as a bilinear form on $\mathscr{E}'^0(X) \times \mathscr{E}'^0(X)$ (let us recall that $\mathscr{E}'^0$ is its own conjugate space), H belongs to $\mathscr{E}^0(X) \hat{\otimes}_\varepsilon \mathscr{E}^0(X)$. But this space can be identified with $\mathscr{E}^0(X \times X)$, and the continuous function thus identified with H is precisely A. It remains for us to see that, A being separately continuous and of positive type, its continuity on the diagonal $X \times X$ implies its continuity on $X \times X$. Now, first of all it is locally bounded on the diagonal, hence locally bounded on $X \times X$, by the Cauchy-Schwarz inequality $|A(x,\xi)| \leqq (A(x,x))^{1/2} (A(\xi,\xi))^{1/2}$. Then, when ξ converges to ξ_0, $A(\cdot,\xi)$ converges weakly to $A(\cdot,\xi_0)$ in $\mathscr{H}$, but with the convergence of the norm $(A(\xi,\xi))^{1/2}$ to the norm $(A(\xi_0,\xi_0))^{1/2}$, hence $A(\cdot,\xi)$ converges strongly to $A(\cdot,\xi_0)$ in $\mathscr{H}$ and consequently in $\mathscr{E}^0$, which expresses exactly that A is continuous on $X \times X$.

□

Corollary 1. *If a sum of functions of positive type on $X \times X$ is separately continuous and locally bounded (resp. continuous), each of the functions is also locally bounded (resp. continuous).*

This is because if a sum of Hilbert subspaces $\mathscr{H}$ of $\mathbb{C}^X$ is a Hilbert subspace of $\mathscr{E}^0(X)$ (resp. has a compact inclusion in $\mathscr{E}^0(X)$), each of the subspaces is also a Hilbert subspace of $\mathscr{E}^0(X)$ (resp. has a compact inclusion in $\mathscr{E}^0(X)$), with the inclusion $\mathscr{H}_i \to \sum_j \mathscr{H}_j$ being continuous. We could develop analogous corollaries to the following propositions; we leave it to the reader to do that: If $\mathscr{A}$ is a locally convex quasi-complete subspace of $\mathbb{C}^X$, with a continuous inclusion, and if a sum of functions of positive type on $X \times X$ belongs to $\mathscr{L}^+(\mathscr{A})$, the same holds for each of the functions in the sum.

Remark. It is not certain that we can replace (9.13) by a double integral, because A, separately continuous and locally bounded, is not necessarily integrable nor even measurable with respect to the product measure $d\mu(x)d\nu(\xi)$. The fact that the integrals (9.13) make sense and are equal is not obvious a priori, and perhaps comes from the fact that A is of positive type.

But let us suppose that X is metrisable on all of its compact subsets, and hence separable on all of its compact subsets. Then the space $\mathscr{C}(K)$ of continuous functions on a compact subset K of X is also separable. Let A be any function on $X \times X$, separately continuous and locally bounded. The map $x \mapsto A(x, \cdot)$ from K into $\mathscr{C}(K)$ with the weak topology is continuous [indeed, when x tends to x_0, $A(x, \cdot)$ converges pointwise to $A(x_0, \cdot)$ and stays bounded on K, so that, as K is metrisable, the dominated convergence theorem of Lebesgue shows that, for every Radon measure ν on K, $\langle A(x, \cdot), \nu\rangle$ converges to $\langle A(x_0, \cdot), \nu\rangle$, so $A(x, \cdot)$ converges weakly to $A(x_0, \cdot)$ in $\mathscr{C}(K)$]. Then let θ be a non-negative Radon measure on $K \times K$, and $\tilde{\theta}$ its image on K under the projection $(x, \xi) \mapsto x$. The map $x \mapsto A(x, \cdot)$ is a fortiori weakly measurable with respect to $\tilde{\theta}$, so strongly measurable since $\mathscr{C}(K)$ is a separable Banach space. This proves that, for any $\delta > 0$, there exists a compact subset K_δ of K such that $\theta(K - K_\delta) \leqq \delta$ and such that $x \mapsto A(x, \cdot)$ is continuous from K_δ into $C(K)$. Then $(x, \xi) \mapsto A(x, \xi)$ is continuous on $K_\delta \times K$; and $\theta((K \times K) - (K_\delta \times K)) = \tilde{\theta}(K - K_\delta) \leqq \delta$; as δ is arbitrary, this proves that A is θ-measurable on $K \times K$; as K and θ are arbitrary, this proves that A is measurable with respect to every Radon measure on $X \times X$.

We have used the fact that A was separately continuous and locally bounded, but only the first hypothesis is necessary to obtain the measurability of A; indeed, it implies that, for every integer $n \geqq 0$, $\inf(n, \sup(-n, \mathrm{Re}A))$, separately continuous and bounded, is measurable, hence so are $\mathrm{Re}A$ and $\mathrm{Im}(A)$ by passing to the limit, and hence A is separately continuous and bounded.

We then see that A is integrable with respect to $d\mu(x)d\bar{\nu}(\xi)$; the integrals (9.13) can be written as

$$\int\int_{X \times X} A(x, \xi) d\mu(x) \overline{d\nu(\xi)} \tag{9.21}$$

and are a priori equal by Fubini.

What's more, according to the two expressions on the right-hand side of (9.13), we see that they define a separately weakly continuous sesquilinear form on $\mathscr{E}'^0(X) \times \mathscr{E}'^0(X)$; so H defined by (9.12) is weakly continuous from $\mathscr{E}'^0(X)$ into $\mathscr{E}^0(X)$. The hypothesis of local metrisability of X thus removes all mysteriousness of the obtained results, and makes them decidedly independent of the fact that A is of positive type.

Let us suppose in particular that X is an open subset of $\mathbb{R}^n$. Then $\mathscr{E}^0(X) \subset \mathscr{D}'(X)$; so H, the kernel $\mathscr{E}'^0(X) \to \mathscr{E}^0(X)$ of $\mathscr{H}$ relative to $\mathscr{E}^0(X)$, gives, by the restriction $\mathscr{D}(X) \to \mathscr{D}'(X)$, the kernel of $\mathscr{H}$ relative to $\mathscr{D}'(X)$ (it will suffice, in Equations (9.12) to (9.14), to replace $d\mu(x)$ and $d\nu(\xi)$ by $\phi(x)dx$ and $\psi(\xi)d\xi$); this kernel is a distribution $H_{x,\xi}$ on $X \times X$ (Example 3 on page 24): this distribution is what is identified with the measurable (with respect to the Lebesgue measure $dxd\xi$) and locally bounded function A. We can thus restate Proposition 24 by saying:

Corollary 2. *For a Hilbert subspace $\mathscr{H}$ of $\mathscr{D}'(X)$, where X is an open subset of $\mathbb{R}^n$, to be a Hilbert subspace of $\mathscr{E}^0(X)$, it is necessary and sufficient that its kernel $H_{x,\xi} \in \mathscr{D}'(X \times X)$ is a separately continuous and locally bounded function A.*

Proof. We just saw that the condition is necessary. It is sufficient according to Corollary 3 of Proposition 21, applied to $E = \mathscr{D}'$ and $G = \mathscr{E}^0$ because, if A is separately continuous and locally bounded, we just saw that H is weakly continuous from $\mathscr{E}'^0(X)$ into $\mathscr{E}^0(X)$, and it is not necessary to assume its positivity, because, by regularisation, every $\mu \in \mathscr{E}'^0(X)$ is the weak limit of a sequence $\mu * \rho_\nu$ of functions in $\mathscr{D}(X)$. □

Proposition 25. *Let X be an open subset of $\mathbb{R}^n$, and let $\mathscr{E}^m(X)$ be the space of functions of class C^m on X, equipped with the topology of uniform convergence on every compact subset with derivatives of all orders up to and including m (of all orders if $m = \infty$, a valid convention for the following). Let $\mathscr{H}$ be a Hilbert subspace of $\mathbb{C}^X$, and A its reproducing kernel of Aronszajn. For $\mathscr{H}$ to be a Hilbert subspace of $\mathscr{E}^m(X)$, it is necessary for A to have separately continuous and locally bounded derivatives (in the usual sense and in the sense of distributions) of orders up to and including m in x and of orders up to and including m in ξ; and it suffices for $\sum_{|p|=m} D_x^p D_\xi^p A$, or $(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial \xi_i})^m A$ (in the sense of distributions) to be a separately continuous and locally bounded function. In this case, let H be the kernel of $\mathscr{H}$ relative to $\mathscr{E}^m(X)$. For $T \in \mathscr{E}'^m(X)$ (distribution of order $\leqq m$), HT is the function of class C^m defined by*

$$(HT)(x) = \int_X A(x,\xi) T(\xi) d\xi \tag{9.21b}$$

and its derivatives of orders up to and including m can be calculated by differentiation under the sign $\int_X$. For this $T \in \mathscr{E}'^m(X)$, we have:

$$\begin{aligned} \langle H\bar{T}, S\rangle &= \int_X S(x)dx \int_X A(x,\xi)\overline{T(\xi)}d\xi \\ &= \int_X \bar{T}(\xi)d\xi \int_X A(x,\xi)S(x)dx. \end{aligned} \tag{9.22}$$

In particular:

$$\langle H\bar{T}, T\rangle = \int_X T(x)dx \int_X A(x,\xi)\overline{T(\xi)}d\xi \geqq 0. \tag{9.23}$$

For the inclusion of $\mathscr{H}$ in $\mathscr{E}^m(X)$ to be compact, it is necessary and sufficient that A has continuous derivatives of order $\leqq m$ in x and $\leqq m$ in ξ (in the usual sense or in the sense of distributions) on $X \times X$.

Proof. Let us first say what we mean by: "A has partial derivatives in the usual sense of orders $\leqq m$ in x and of orders $\leqq m$ in ξ". We mean that we can apply, *in whatever order*, partial differentiation with respect to $x_1, x_2, ..., x_n, \xi_1, \xi_2, ..., \xi_n$, provided that the number of times we differentiate with respect to the x_i is $\leqq m$ and that the number of times we differentiate with respect to the ξ_i is $\leqq m$; and moreover that the obtained result *is independent of the order of differentiation*[46].

Let us recall that if a function in one variable has a derivative *everywhere*, which is a locally integrable function, then it is the indefinite integral of it. We easily deduce from this that if a locally integrable function on $X \times X$ has a partial derivative $\frac{\partial}{\partial x_i}$ or $\frac{\partial}{\partial \xi_i}$ *everywhere*, and if this derivative is locally integrable, it is also the derivative in the sense of distributions[47].

Thus, if A has partial derivatives of order $\leqq m$ in x and $\leqq m$ in ξ, in the usual sense, which are separately continuous and locally bounded, these are also its derivatives in the sense of distributions. Two derivatives differing only by the order of differentiation are then equal as distributions, so almost everywhere equal, so everywhere equal by separate continuity.

1°) Let us suppose that $\mathscr{H}$ is a Hilbert subspace of $\mathscr{E}^m(X)$. Then, for every $x \in X$, $A(\cdot, x)$ is in $\mathscr{E}^m(X)$, and thus so is $A(x, \cdot)$ by Hermitian symmetry. Then the right-hand side of (9.21b), for fixed x, makes sense, and represents a linear form in T, weakly continuous on $\mathscr{E}'^m$. As H is weakly continuous and linear from $\mathscr{E}'^m$ into $\mathscr{E}^m$, the left-hand side has the same properties. They coincide for $T = \delta_{(\xi)}$, by the very definition of the reproducing kernel; the point masses form a weakly total set in $\mathscr{E}'^m$, so the equality (9.21) is true. The right-hand side is then necessarily a function in x of class C^m; by applying the distribution $S \in \mathscr{E}'^m$ to both sides, we obtain the first equality in (9.22). The second can be deduced by Hermitian symmetry (and by exchanging x and ξ); from this, (9.23) is immediate.

This proof is also valid for $m = 0$, and gives a variant of part 3°) of the preceding Proposition.

2°) Let us assume that $\mathscr{H}$ is a Hilbert subspace of $\mathscr{E}^m(X)$. Let us apply (9.22) for

$$S = (D^p\delta)_{(x)}, T = (D^q\delta)_{(\xi)}. \qquad \text{We obtain:}$$

$$(9.24) \quad \begin{aligned} &\langle H((D^q\delta)_{(\xi)}), (D^p\delta)_{(x)}\rangle \\ &= (-1)^{|p+q|} D_x^p D_\xi^q A(x, \xi) \\ &= (-1)^{|p+q|} D_x^q i D_x^p A(x, \xi). \end{aligned}$$

So firstly these two derivatives exist, and $D_\xi^q A(\cdot, \xi)$ is of class C^m in x for fixed ξ, and $D_x^p A(x, \cdot)$ of class C^m in ξ for fixed x. And then $\xi \mapsto (D^q\delta)_{(\xi)}$

[46] Independence of the order of differentiation is not automatic, because we do not assume the continuity of the considered derivatives on $X \times X$.

[47] Schwartz [1], Chapter II, Theorem V.

and $x \mapsto (D^p\delta)_{(x)}$ are continuous from X into $\mathscr{E}'^m(X)$ with the weak topology; as $\tilde{H}$ is separately weakly continuous on $\mathscr{E}'^m(X) \times \mathscr{E}'^m(X)$, $D_x^p D_\xi^q A = D_\xi^q D_x^p A$ is separately continuous. If x and ξ range over compact subsets of X, $(D^p\delta)_{(x)}$ and $(D^q\delta)_{(\xi)}$ range over the bounded subsets of $\mathscr{E}'^m(X)$, so $D_x^p D_\xi^q A(x,\xi)$ stays bounded; it is locally bounded.

This does not show completely that the derivatives of orders $\leqq m$ in x and $\leqq m$ in ξ of A exist (with permutability of order of differentiation); we only saw that $D_x^p D_\xi^q A$ and $D_\xi^q D_x^p A$ exist for $|p| \leqq m$, $|q| \leqq m$, and that they are equal. It is, however, obvious for $m = 0$ (and true also for $m = 1$). Let us assume that it has been shown for $m = 0, 1, 2, ..., l-1$, and let us show it for $m = l$. A derivative of order $\leqq l$ in x and $\leqq l$ in ξ of A is of the form $D_x^\alpha D_\xi^\beta D'$, or $D_\xi^\beta D_x^\alpha D'$, where D' is a partial derivative of order $\leqq l-1$ in x and $\leqq l-1$ in ξ. Let us first consider the case $D = D_x^\alpha D_\xi^\beta D'$. Since $\mathscr{H}$ is a Hilbert subspace of $\mathscr{E}^l$, it is a fortiori a Hilbert subspace of $\mathscr{E}^{l-1}$; so, if D' is of total index p' in x and of total index q' in ξ, the induction hypothesis says that $D'A$ exists, and is also given by $D_x^{p'} D_\xi^{q'} A$ and $D_\xi^{q'} D_x^{p'} A$. Then $D_\xi^\beta D'A$ exists by the previous results and is given by $D_\xi^{\beta+q'} D_x^{p'} A$; it is thus also equal to $D_x^{p'} D_\xi^{\beta+q'} A$, so that DA exists and is equal to $D_x^{\alpha+p'} D_\xi^{\beta+q'} A$; and it is also equal to $D_\xi^{\beta+q'} D_x^{\alpha+p'} A$. The second case $D = D_\xi^\beta D_x^\alpha D'$ follows the same reasoning, which proves the property in the statement.

Remark. To complete the obtained formulae, it is good to add the following: by applying (9.21b) to $T = D^p\delta_{(a)}$, with $a \in X$, we see that

(9.25) $$(HD^p\delta_{(a)})(x) = (-1)^{|p|}(D_\xi^p A)(x,a);$$

so $D_\xi^p A(\cdot, a) \in \mathscr{H}$[48].

Equations (9.22) and (9.23) then give:

(9.26) $$(D_\xi^q A(\cdot,b) \mid D_\xi^p A(\cdot,a)) = D_x^p D_\xi^q A(a,b)$$
$$\left\| D_\xi^p A(\cdot,a) \right\|_{\mathscr{H}}^2 = D_x^p D_\xi^p A(a,a).$$

Finally, for $h \in \mathscr{H}$, (4.2) gives

(9.27) $$(D^p h)(a) = \langle h, (-1)^{|p|} D^p \delta_{(a)} \rangle = (h \mid D_\xi^p A(\cdot,a))_{\mathscr{H}}$$

so in particular the upper bound

(9.28) $$|(D^p h)(a)| \leqq \|h\|_{\mathscr{H}} \, ((D_x^p D_\xi^p A)(a,a))^{1/2}.$$

There is no reason, in these formulae, to restrict ourselves to differential operators with constant coefficients. Let P and Q be two differential

[48] Such a notation $D_\xi^p A(\cdot,a)$ shows the derivative of A with respect to the second variable, considered, for this second variable being equal to a, as a function of the first variable. We could also write it, more rigorously, as $x \mapsto (D_\xi^p A(x,\xi))_{\xi=a}$.

operators with any C^∞ coefficients of order $\leqq m$. By applying (9.21b) to ${}^tP\delta_{(a)}$ we obtain

$$H \cdot {}^tP\delta_{(a)} = \int_X A(\cdot,\xi)\, {}^tP\delta_{(a)} d\xi = P_\xi A(\cdot, a). \tag{9.30}$$

(9.22) and (9.23) then give

$$\begin{aligned} &(Q_\xi A(\cdot,b) \mid P_\xi A(\cdot,a))_{\mathscr{H}} \\ &= \langle Q_\xi A(\cdot,b), \overline{{}^tP}\delta_{(a)}\rangle \\ &= \overline{P}_x Q_\xi A(a,b). \end{aligned} \tag{9.31}$$

For $h \in \mathscr{H}$, (4.2) gives

$$\begin{aligned} \overline{P}h(a) &= \langle h, \overline{{}^tP}\delta_{(a)}\rangle \\ &= (h \mid P_\xi A(\cdot,a))_{\mathscr{H}}. \end{aligned} \tag{9.32}$$

3°) $D^p_x D^p_\xi H$ (distribution) is the kernel of $D^p\mathscr{H}$ (Example 3, page 43); so $\sum\limits_{|p|=m} D^p_x D^p_\xi H$ is the kernel of $\sum\limits_{|p|=m} D^p\mathscr{H}$. But, according to the closed graph theorem, $\mathscr{H}$ is a Hilbert subspace of $\mathscr{E}^m(X)$ as soon as it is contained in $\mathscr{E}^m(X)$; for that, it is necessary and sufficient that $\sum\limits_{|p|=m} D^p\mathscr{H}$ is contained in $\mathscr{E}^0(X)$, or is a Hilbert subspace of $\mathscr{E}^0(X)$, and for that it is necessary and sufficient that its kernel $\sum\limits_{|p|=m} D^p_x D^p_\xi H$ is a separately continuous and locally bounded function, which shows the first sufficient condition.

The second can be shown by induction on m. It is true for $m = 0$ (or for $m = 1$, since it then coincides with the first). Let us suppose that it has been shown for $m = 0, 1, 2, ..., l-1$, and let us show it for $m = l$. For $\mathscr{H}$ to be a Hilbert subspace of $\mathscr{E}^l$, it is necessary and sufficient that $\sum\limits_{i=1}^n \frac{\partial}{\partial x_i}\mathscr{H}$ is a Hilbert subspace of $\mathscr{E}^{l-1}$; as the kernel of $\mathscr{E}^{l-1}$ is $\sum\limits_{i=1}^n \frac{\partial^2 H}{\partial x_i \partial \xi_i}$, it is necessary and sufficient, by the induction hypothesis, that $\left(\sum\limits_{i=1}^n \frac{\partial^2}{\partial x_i \partial \xi_i}\right)^{l-1} \left(\sum\limits_{i=1}^n \frac{\partial^2}{\partial x_i \partial \xi_i}\right) H$ is a separately continuous and locally bounded function, which is the result we were after.

4°) By Proposition 9c, for the inclusion of $\mathscr{H}$ in $\mathscr{E}^m(X)$ to be compact, it is necessary and sufficient that A is in $\mathscr{E}^m(X)\hat{\otimes}_\varepsilon\mathscr{E}^m(X)$ and that $\tilde{H}$ is a non-negative sesquilinear form on $\mathscr{E}'^m(X) \times \mathscr{E}'^m(X)$. The non-negativity of $\tilde{H}$ is automatic once A is in $\mathscr{E}^m(X)\hat{\otimes}_\varepsilon\mathscr{E}^m(X)$ and A is of positive type, following the criterion in Corollary 3 of Proposition 21: every $T \in \mathscr{E}'^m$ is a weak limit of a sequence of discrete measures, so belong to $(\mathbb{C}^X)'$. Then $\mathscr{E}^m(X)\hat{\otimes}_\varepsilon\mathscr{E}^m(X) = \mathscr{E}^{m,m}_{x,\xi}$ is precisely the space of functions whose derivatives of order $\leqq m$ in x and $\leqq m$ in ξ are continuous on $X \times X$.

□

Remark. If $\mathscr{H} \in \mathscr{E}^m(X)$, there is a compact inclusion into $\mathscr{E}^{m-1}(X)$, since the inclusion $\mathscr{E}^m(X) \to \mathscr{E}^{m-1}(X)$ transforms every bounded subset into relatively compact subsets. In addition, every function on $X \times X$ with locally bounded derivatives of order $\leqq m$ in x and in ξ has continuous derivatives of order $\leqq m-1$ in x and in ξ on $X \times X$ (and even locally Lipschitz). This is compatible with 4°).

Proposition 26. *Let X be a locally compact space for $m = 0$, and an open subset of $\mathbb{R}^n$ for $m \geqq 1$ (potentially infinite). For a Hilbert subspace $\mathscr{H}$ of $\mathscr{E}^m(X)$ to be a Hilbert subspace of $\mathscr{B}^m(X)$ (space of continuous bounded functions on X, with derivatives of order $\leqq m$ also continuous and bounded, equipped with the topology of uniform convergence on X of each derivative of order $\leqq m$), it is necessary that each derivative $D_x^p D_\xi^q A$ for $|p| \leqq m$ and $|q| \leqq m$ is bounded on $X \times X$, and it is sufficient that $\sum_{|p| \leqq m} D_x^p D_\xi^p A$ is bounded on the diagonal of $X \times X$.*

Proof. Let us first suppose that $\mathscr{H}$ is a Hilbert subspace of $\mathscr{B}^m(X)$. For $|p| \leqq m$, $D^p \delta_{(x)}$ ranges over a weakly bounded subset of $\mathscr{B}^m(X)$ as x ranges over X. As the sesquilinear form $\tilde{H}$ associated to H stays bounded on every product of bounded subsets of $\mathscr{B}^m(X)$, Equation (9.24) shows that $D_x^p D_\xi^q A$ is bounded on $X \times X$.

Conversely, let us suppose that A is bounded on the diagonal of $X \times X$, so bounded on $X \times X$ by $|A(x,\xi)| \leqq (A(x,x))^{1/2}(A(\xi,\xi))^{1/2}$. Equation (9.28) for $|p| = 0$ shows that each h in $\mathscr{H}$ is bounded on X, so $\mathscr{H}$ is a Hilbert subspace of $\mathscr{B}^0(X)$ by the closed graph theorem. If $\sum_{|p| \leqq m} D_x^p D_\xi^p A$ is bounded on $X \times X$, as it is the kernel of $\sum_{|p| \leqq m} D^p \mathscr{H}$, this proves that each $D^p \mathscr{H}$ is in $\mathscr{B}^0(X)$, so $\mathscr{H}$ is in $\mathscr{B}^m(X)$ and $\mathscr{H}$ is a Hilbert subspace of $\mathscr{B}^m(X)$ by the closed graph theorem. □

Proposition 27. *Let X be a locally compact space for $m = 0$, and an open subset of $\mathbb{R}^n$ for $m \geqq 1$. Let $\mathscr{U}$ be a uniform structure on X, compatible with its topology. Let $\mathscr{B}^m_{\mathscr{U}}(X)$ be the (complete) space of bounded and uniformly continuous (with respect to $\mathscr{U}$) functions on X, each of whose derivatives of order $\leqq m$ are also bounded and uniformly continuous with respect to $\mathscr{U}$, equipped with the topology induced by $\mathscr{B}^m(X)$. For a Hilbert subspace $\mathscr{H}$ of $\mathscr{B}^m(X)$ to be a Hilbert subspace of $\mathscr{B}^m_{\mathscr{U}}(X)$, it is necessary for each function $(D_x^p D_\xi^q A)(\cdot, a)$, $|p| \leqq m$, $|q| \leqq m$, $a \in X$, to be uniformly continuous with respect to $\mathscr{U}$, and it is sufficient for each function $D_x^p(A)(\cdot, a)$ to be so.*

Proof. If $\mathscr{H} \subset \mathscr{B}^m_{\mathscr{U}}$, the function $D_\xi^q A(\cdot, a) \in \mathscr{H}$ is in $\mathscr{B}^m_{\mathscr{U}}$ so each of its derivatives $D_x^p D_\xi^q A(\cdot, a)$ is uniformly continuous on X with respect to $\mathscr{U}$.

Conversely, if each $D_x^p A(\cdot, a)$ is uniformly continuous on X with respect to $\mathscr{U}$, then $A(\cdot, a)$ is in $\mathscr{B}^m_{\mathscr{U}}$; every $h \in \mathscr{H}$ is the limit, in $\mathscr{H}$ and hence in $\mathscr{B}^m$, of finite combinations of functions $A(\cdot, a) \in \mathscr{B}^m_{\mathscr{U}}$, and $\mathscr{B}^m_{\mathscr{U}}$ is closed in $\mathscr{B}^m$; so $h \in \mathscr{B}^m_{\mathscr{U}}$, and $\mathscr{H}$ is a Hilbert subspace of $\mathscr{B}^m_{\mathscr{U}}$. □

Remark. This type of statements, pertaining to $\mathscr{B}^m$ and $\mathscr{B}^m_{\mathscr{U}}$, is found each time one has a locally convex, quasi-complete Hausdorff vector space and a closed subspace, equipped with the induced topology.

Application. Bergman's invariant Hermitian form on a complex analytic manifold. [49]

Let V be a complex analytic manifold, of complex dimension n. Let $\overset{n}{\mathscr{H}}$ be the Hilbert space of holomorphic and square-integrable differential forms of degree n, equipped with the scalar product:

$$(9.33)\qquad (\omega \mid \bar{\omega})_{\mathscr{H}} = \left(\frac{1}{2i}\right)^n \varepsilon \int_V \omega \wedge \bar{\omega}.$$

The form ω is of type $(n,0)$ and $\bar{\omega}$ of type $(0,n)$ so $\omega \wedge \bar{\omega}$ is of degree $2n$, and the integral makes sense, since ω and $\bar{\omega}$ are square-integrable. In accordance with the established practice, the orientation of V is that for which, in local coordinates $z_1, z_2, ..., z_n$, the form $\left(\frac{1}{2i}\right)^n dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge ... \wedge dz_n \wedge d\bar{z}_n$ is non-negative; $\varepsilon = \pm 1$ is then the sign of the form

$$\left(\frac{1}{2i}\right)^n dz_1 \wedge dz_2 \wedge ... \wedge dz_n \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge ... \wedge d\bar{z}_n,$$

such that, for every ω, $(\omega \mid \omega)_{\mathscr{H}} \geqq 0$, and $\mathscr{H}$ is Hilbert.

$\mathscr{H}$ is a Hilbert subspace of the space $\mathscr{E}$ of the C^∞ differential forms of type $(n,0)$[50]; it thus has an associated kernel H, a continuous linear map from $\left(\overline{\overset{n,0}{\mathscr{E}}}\right)' = \overline{\overset{0,n}{\mathscr{E}'}} = \overset{n,0}{\mathscr{E}'}$ into $\overset{n,0}{\mathscr{E}}$.

We define the anti-duality between $\overset{n,0}{\mathscr{E}}$ and $\overset{n,0}{\mathscr{E}'}$ by

$$(9.33\text{b})\qquad (f \mid T)_{\overset{n,0}{\mathscr{E}}, \overset{n,0}{\mathscr{E}}} = \left(\frac{1}{2i}\right)^n \varepsilon \int f \wedge \bar{T} = \overline{(T \mid f)}_{\overset{n,0}{\mathscr{E}}, \overset{n,0}{\mathscr{E}}}.$$

We will identify H with a C^∞ differential form of type $(n,0,0,n)$ on $V \times V$, by the following convention:

$$(9.34)\qquad H \cdot T = \left(\frac{1}{2i}\right)^n (-1)^n \varepsilon \int_V H_{x,\xi} \wedge T_\xi.$$

Let Ω be an open subset of V, equipped with local coordinates z_i. Let i be the natural inclusion $\mathscr{D}(\Omega) \to \mathscr{D}(V)$; its adjoint i^* is the restriction operation $\rho : \mathscr{D}'(V) \to \mathscr{D}'(\Omega)$, which associates its restriction to Ω to each current of V. The restriction of H to Ω is nothing but the composition map $\rho H i$; according to Proposition 21, it is thus the kernel associated to the Hilbert space image $\rho(\mathscr{H})$ of $\mathscr{H}$ in $\overset{n,0}{\mathscr{D}'}(\Omega)$; an element of $\rho(\mathscr{H})$ is a holomorphic form of degree n on Ω, which is the restriction of a square-integrable holomorphic form on V. On $\Omega \times \Omega$, the differential form H is given by:

$$(9.35)\qquad A(x,\xi) dz_1 \wedge dz_2 \wedge ... \wedge dz_n \wedge d\zeta_1 \wedge d\zeta_2 \wedge ... \wedge d\zeta_n,$$

where A is a C^∞ function. Let $T \in \overset{0,0}{\mathscr{E}'}(\Omega)$; then $T d\zeta_1 \wedge d\zeta_2 \wedge ... \wedge d\zeta_n$ is an element of $\overset{n,0}{\mathscr{E}'}(\Omega)$, and, taking into account the fact that $d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_2 \wedge ... \wedge d\zeta_n \wedge d\bar{\zeta}_n$

[49] See Bergman [1].

[50] See, for example, André Weil [1], beginning of Chapter I.

is $(2i)^n$ times the Lebesgue measure $d\xi$ on Ω, we see that

$$H \cdot (T d\zeta_1 \wedge d\zeta_2 \wedge ... \wedge d\zeta_n) = \left(\int_V A(x,\xi) T_\xi d\xi \right) dz_1 \wedge dz_2 \wedge ... \wedge dz_n.$$

So if we agree to identify, on Ω, the forms of type $(n,0)$ with the functions, or the $(n,0)$-currents with the distributions, by identifying T with $\tilde{T} = T dz_1 \wedge dz_2 \wedge ... \wedge dz_n$, $\rho(\mathscr{H})$ is identified with the space of holomorphic functions on Ω, whose product with $dz_1 \wedge dz_2 \wedge ... \wedge dz_n$ is the restriction of a square-integrable holomorphic form on V; and its kernel $\mathscr{E}'(\Omega) \to \mathscr{E}(\Omega)$ is the reproducing function $(x,\xi) \mapsto A(x,\xi)$. We thus have, in particular, $A(x,x) \geqq 0$. Moreover, we have $\frac{\partial}{\partial \bar{z}_i} \rho(\mathscr{H}) = 0$ so, according to Corollary 2 of Proposition 21, $\frac{\partial A}{\partial \bar{z}_i} = \frac{\partial A}{\partial \zeta_i} = 0$: A is holomorphic in x and anti-holomorphic in ξ. The form H thus has the same property globally.

Let us suppose that $A(x,x) > 0$; the function $A(x,\cdot)$ is thus non-negative in x; as it is in $\rho(\mathscr{H})$, there exists a form in $\mathscr{H}$ which is non-zero in x. Conversely, let us suppose that this condition is realised; (9.11b) shows that $A(x,x)$ is also non-zero. *We will henceforth assume that, for every x in V, there exists a form in $\mathscr{H}$ which is non-zero in x*; then, for every local chart, $A(x,x) > 0$ everywhere.

Let K be the restriction of the form H to the diagonal of $V \times V$, multiplied by $\varepsilon \left(\frac{1}{2i}\right)^n$; by identifying this diagonal to V, it is a differential form of degree $2n$ on V; on Ω it is given by

$$K = A(x,x)dx = B(x)dx, \tag{9.37}$$

an everywhere strictly positive C^∞ differential form.

But, on a complex analytic manifold V, a real, everywhere $\neq 0$, C^∞ differential form of degree $2n$ allows for the definition of a Hermitian form. Let us indeed write:

$$g = \sum_{i,j} g_{i,j} dz_i d\bar{z}_j = \sum_{i,j} \frac{\partial^2 \log B}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j. \tag{9.38}$$

This form is independent of the system of local coordinates z_i, chosen in Ω. If, indeed, z'_i is another, we first have

$$\begin{aligned} K &= \varepsilon \left(\frac{1}{2i}\right)^n B dz_i \wedge dz_2 \wedge ... \wedge dz_n \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge ... \wedge d\bar{z}_n \\ &= \varepsilon \left(\frac{1}{2i}\right)^n B' dz'_1 \wedge dz'_2 \wedge ... \wedge dz'_n \wedge d\bar{z}'_1 \wedge d\bar{z}'_2 \wedge ... \wedge d\bar{z}'_n, \end{aligned} \tag{9.39}$$

whence

$$B = B' J \bar{J}, \tag{9.40}$$

where J is a the Jacobian determinant of the z_i with respect to z'_j.

Then

$$\frac{\partial^2 \log B}{\partial z_i \partial \bar{z}_j} = \frac{\partial^2 \log B'}{\partial z_i \partial \bar{z}_j} + \frac{\partial^2 \log J}{\partial z_i \partial \bar{z}_j} + \frac{\partial^2 \log \bar{J}}{\partial z_i \partial \bar{z}_j} = \frac{\partial^2 \log B'}{\partial z_i \partial \bar{z}_j}, \tag{9.41}$$

because $\frac{\partial}{\partial \bar{z}_j} \log J = \frac{\partial}{\partial z_i} \log \bar{J} = 0$.

After that, we immediately have

$$\frac{\partial^2 \log B'}{\partial z_i \partial \bar{z}_j} = \frac{\partial}{\partial z_i} \sum_l \left(\frac{\partial \log B'}{\partial \bar{z}'_l} \overline{\frac{\partial z'_l}{\partial z_j}} \right) = \sum_l \left(\frac{\partial}{\partial z_i} \frac{\partial \log B'}{\partial \bar{z}'_l} \right) \overline{\frac{\partial z'_l}{\partial z_j}}$$
$$= \sum_{k,l} \frac{\partial^2 \log B'}{\partial z'_k \partial \bar{z}'_l} \frac{\partial z'_k}{\partial z_i} \overline{\frac{\partial z'_l}{\partial z_j}}. \tag{9.42}$$

We thus have

$$\sum_{i,j} \frac{\partial^2 \log B}{\partial z_i \partial \bar{z}_j} dz_i \overline{dz_j} = \sum_{i,j} \frac{\partial^2 \log B'}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$
$$= \sum_{i,j,k,l} \frac{\partial^2 \log B'}{\partial z'_k \partial \bar{z}'_l} \left(\frac{\partial z'_k}{\partial z_i} dz_i \right) \overline{\left(\frac{\partial z'_l}{\partial z_j} dz_j \right)} = \sum_{k,l} \frac{\partial^2 \log B'}{\partial z'_k \partial \bar{z}'_l} dz'_k d\bar{z}'_l. \tag{9.42b}$$

The form defined on Ω is thus intrinsic, and we have a form on V, which is manifestly C^∞, and Hermitian, because B is real. Let us now show that it is non-negative. We have to show that, for any $a \in V$ and any complex numbers Z_i, we have:

$$\sum_{i,j} \frac{\partial^2 \log B}{\partial z_i \partial \bar{z}_j}(a) Z_i \bar{Z}_j \geqq 0. \tag{9.43}$$

Let X be a field of holomorphic vectors, $X = \sum_i X_i \frac{\partial}{\partial z_i}$ such that $X_i(a) = Z_i$. Taking into account:

$$g_{i,j} = \frac{\partial^2 \log B}{\partial z_i \partial \bar{z}_j} = -\frac{1}{B^2} \frac{\partial B}{\partial z_i} \frac{\partial B}{\partial \bar{z}_j} + \frac{1}{B} \frac{\partial^2 B}{\partial z_i \partial \bar{z}_j}, \tag{9.44}$$

the preceding inequality can also be written under the form

$$\sum_{i,j} Z_i \bar{Z}_j \frac{\partial B}{\partial z_i}(a) \frac{\partial B}{\partial \bar{z}_j}(a) \leqq B(a) \sum_{i,j} Z_i \bar{Z}_j \frac{\partial^2 B}{\partial z_i \partial \bar{z}_j}(a). \tag{9.45}$$

But, as A is holomorphic in x and anti-holomorphic in ξ, we have

$$\frac{\partial B}{\partial z_i}(a) = \left(\frac{\partial A}{\partial z_i} + \frac{\partial A}{\partial \zeta_i} \right)(a, a) = \frac{\partial A}{\partial z_i}(a, a)$$

and likewise:

$$\frac{\partial B}{\partial \bar{z}_i}(a) = \frac{\partial A}{\partial \bar{\zeta}_i}(a, a), \qquad \frac{\partial^2 B}{\partial z_i \partial \bar{z}_j}(a) = \frac{\partial^2 A}{\partial z_i \partial \bar{\zeta}_j}(a, a).$$

(9.45) thus can be written as

$$\left| \sum_i Z_i \frac{\partial A}{\partial z_i}(a, a) \right|^2 \leqq A(a, a) \sum_{i,j} Z_i \bar{Z}_j \frac{\partial^2 A}{\partial z_i \partial \bar{\zeta}_j}(a, a) \tag{9.46}$$

or

$$\left| \theta(X_x) \cdot A \right|^2 \leqq A(\theta(X_x) \cdot \theta(\bar{X}_\xi) \cdot A) \qquad \text{for } \xi = x = a.$$

Let us apply the formula of reproducing kernels (9.31) for the differential operators 1 and $\theta(X)$. The above is then equivalent to

$$(9.48) \qquad \left| \left(A(\cdot,a) \mid \theta(\bar{X}_\xi) A(\cdot,a) \right)_{\rho(\mathscr{H})} \right|^2 \leqq \left\| A(\cdot,a) \right\|^2_{\rho(\mathscr{H})} \left\| \theta(\bar{X}_\xi) A(\cdot,a) \right\|^2_{\rho(\mathscr{H})},$$

which is simply the Cauchy-Schwarz inequality. In general, the inequality will be strict, and $\sum_{i,j} g_{i,j}(a) Z_i \bar{Z}_j > 0$. We will have the equality $= 0$ if and only if there exists a complex number λ such that

$$\theta(\bar{X}_\xi) A(\cdot,a) - \bar{\lambda} A(\cdot,a) = 0.$$

By (9.32), it is equivalent to saying that, for every $h \in \rho(\mathscr{H})$, we have $(\theta(X) \cdot h)(a) - \lambda h(a) = 0$ or

$$\sum_i Z_i \frac{\partial h}{\partial z_i}(a) - \lambda h(a) = 0.$$

So the form g will be positive definite if and only if, for every a and every system of local coordinates z_i in the neighbourhood of a, there exists no non-trivial linear relationship between the values of ω and its derivatives $\frac{\partial \omega}{\partial z_i}$ in a, satisfied by every ω in $\mathscr{H}$ (i.e. the $n+1$ linear forms on $\mathscr{H}$: $\omega \mapsto \omega(a)$, $\omega \mapsto \frac{\partial \omega}{\partial z_i}(a)$, are independent).

We can canonically associate a differential form γ of type $(1,1)$ to the Hermitian form g, which will be, in Ω:

$$(9.49) \qquad \gamma = \sum_{i,j} g_{i,j} dz_i \wedge d\bar{z}_j = d_z d_{\bar{z}} \log B.$$

The expression of this form shows that it is closed, so V is a Kähler manifold with respect to g.

(Let us finally remark that, if V does not satisfy the condition indicated on page 62 with regards to the the existence of a square integrable holomorphic form, we can always still construct g, but it has some singularities on the analytic set W of zeros shared by all holomorphic forms of $\mathscr{H}$. We assume that there exists at least one such non-zero form, without which $K = 0$, and we can then take $g = 0$). The function $\log B$ is locally integrable; if, indeed, $\tilde{h} \in \rho(\mathscr{H})$ is not identically zero, the Equation (9.11b) shows that, in Ω:

$$(9.50) \qquad B(x) = \frac{h(x)}{\|h\|};$$

as $\log h$ is locally integrable, $\log B$ is likewise locally integrable. So Equation (9.49) defines a closed current of type (1,1) on Ω; the method employed in Equation (9.41) shows that the definition is intrinsic and thus valid on V.

We can summarise the preceding results as follows:

Proposition 27b. *Let V be a complex analytic manifold, of complex dimension n. Suppose that, for every a in V, and every system of local coordinates z_i in the neighbourhood of a, there exists no non-trivial linear relationship between the values at a of ω and the $\partial\omega/\partial z_i$, satisfied by all holomorphic forms ω of degree n, square integrable on V (example: V is a bounded open subset of $\mathbb{C}^n$). Then Equation (9.38) defines a positive definite Hermitian form g on V, for which V is a Kähler manifold.*

§10. Case where $\bar{E}'$ is a subspace of E

In this section, we will take as given a Hermitian inclusion I of $\bar{E}'$ in E. Through I we will identify $\bar{E}'$ with a subspace of E and we will often call I the identity. By Proposition 4, I is weakly (and strongly) continuous; moreover, as I is injective, $I = I^*$ has a dense image, and $\bar{E}'$ is dense in E. If $I \geqq 0$, it defines a Hilbert subspace of E, containing $\bar{E}'$ as a dense subspace with weakly continuous inclusion; it will be given by the structure and we will call it the canonical L^2 Hilbert subspace of E. The example we will most often consider is $E = \mathscr{D}'(X)$, the space of distributions on an open subset X of $\mathbb{R}^n$, with $\bar{E}' = \mathscr{D} \subset \mathscr{D}'$; L^2 here is indeed the usual space of square integrable (with respect to dx) functions on X. We could also take $E = \mathscr{D}_c'^m(X)$ with $\bar{E}' = \mathscr{D}^m(X)$, $E = \mathscr{S}'(\mathbb{R}^n)$ with $\bar{E}' = \mathscr{S}(\mathbb{R}^n)$, etc. Conversely, if $\mathscr{H}$ is a *dense* Hilbert subspace of a space E, its kernel H is a non-negative inclusion of $\bar{E}'$ in E, weakly continuous, which can play the role of I, and L^2 is then $\mathscr{H}$.

The fact that I is Hermitian yields following: for $\bar{e}' \in \bar{E}'$ and $f' \in E'$, we have

$$\text{(10.1)} \qquad \begin{cases} \langle \bar{e}', f' \rangle_{E,E'} = \overline{\langle \bar{f}', e' \rangle_{E,E'}} \\ (\bar{e}' \mid \bar{f}')_{E,\bar{E}'} = (\bar{e}' \mid \bar{f}')_{\bar{E}',E} \end{cases}$$

the latter scalar products $(\cdot \mid \cdot)$ are those of (0.4), but also $(\cdot \mid \cdot)_{L^2}$ which are induced by L^2. In the case $E = \mathscr{D}'(X)$ there exists a natural anti-involution on $\mathscr{D}$ and $\mathscr{D}'$, $\bar{E}' = E'$ with $\bar{I} = I$, so $I^* = {}^tI = \bar{I} = I$ (that is to say, if $\bar{\phi}$ is the conjugate of ϕ in $\mathscr{D}$, it is also its conjugate in $\mathscr{D}'$) and (10.1) for $\phi, \psi \in \mathscr{D}$ is also given by

$$\text{(10.2)} \qquad \begin{cases} \langle \phi, \psi \rangle_{\mathscr{D}',\mathscr{D}} & = \langle \phi, \psi \rangle_{\mathscr{D},\mathscr{D}'} \\ (\phi \mid \psi)_{\mathscr{D}',\mathscr{D}} = (\phi \mid \psi)_{\mathscr{D},\mathscr{D}'} & = (\phi \mid \psi)_{L^2}. \end{cases}$$

Normal subspace of E.

We will say that a locally convex topological vector subspace F of E, with a weakly continuous inclusion $F \to E$, is *normal*, if $\bar{E}'$ is a dense subspace of F, with a weakly (thus strongly) continuous inclusion.

It is worth knowing that in some cases this condition of continuity of the inclusion $\bar{E}' \to F$ is automatic. This will follow from the Corollary of Proposition 28.

Proposition 28. *Let A be a locally convex, quasi-complete Hausdorff space, and B a locally convex metrisable space. A linear map u from A' into B', continuous with respect to $\sigma(A', A)$ and a locally convex Hausdorff topology $\mathscr{C}$ coarser than $\sigma(B', B)$, is also continuous with respect to $\sigma(A', A)$ and $\sigma(B', B)$.*

Proof. The dual of $B'_{\mathscr{C}}$ equipped with $\mathscr{C}$ is a subspace B_1 of B, weakly dense in B (the inclusion $B_1 \to B$ is the transpose of the inclusion $B' \to B'_{\mathscr{C}}$) so strongly dense in B. The transpose ${}^tu$ of u is linear from B_1 into A, and continuous with respect to $\sigma(B', B)$ and $\sigma(A, A')$. Then the image under ${}^tu$ of a B-bounded subset of B_1 is weakly bounded in A, so strongly bounded; so, as B is metrisable, ${}^tu$ is continuous from B_1 equipped with the topology induced by $\mathscr{C}$, into A. But B_1, being dense in a metrisable space B, is strictly dense, and A is quasi-complete; so ${}^tu$ extends to a continuous linear map $\widehat{{}^tu}$ from B into

A. The transpose of $\widehat{{}^t u}$ is necessarily again u (indeed, for $a' \in A'$ and $b' \in B'$, we have $\langle {}^t\,\widehat{{}^t u}\, a', b\rangle = \langle a', \widehat{{}^t u}\, b\rangle$; by taking $b \in B_1$, we see that $({}^t\,\widehat{{}^t u}\, a' - ua') \in B'$ is orthogonal to B_1 which is dense in B, so vanishes). So u is continuous with respect to $\sigma(A', A)$ and $\sigma(B', B)$. □

Corollary. *With the conditions indicated at the beginning of the section, if F is a subspace of E with weakly continuous inclusion, and if F is a reflexive Fréchet dual, in particular if it is a reflexive Banach space or a Hilbert space, and if $\bar{E}'$ is contained in F, the inclusion $\bar{E}' \to F$ is automatically weakly continuous.*

Proof. It suffices to apply the Proposition to $A = \bar{E}$, $B = F'$ with the strong topology (reflexive Fréchet) and the inclusion $u : \bar{E}' \to F$. Then u is continuous with respect to $\sigma(\bar{E}', \bar{E})$ and $\sigma(F, E')$ (topology induced by $\sigma(E, E')$), since $\bar{E}' \xrightarrow{I} E$ is weakly continuous. But $F \to E$ is weakly continuous, so $\sigma(F, E')$ is coarser than $\sigma(F, F')$. The Proposition says that $\bar{E}' \to F$ is weakly continuous. □

So let F be a normal subspace of E. Then, from the dense weakly continuous inclusions $\bar{E}' \xrightarrow{u} F \xrightarrow{j} E$ we will deduce, by passing to adjoints, the weakly continuous and weakly dense inclusions $\bar{E}' \xrightarrow{j^*} F \xrightarrow{u^*} E$; *we can thus again identify $\bar{F}'$ with the weak topology with a normal subspace of E. If, moreover, F is reflexive,* $\bar{E}'$ will be strongly dense in $\bar{F}'$ and $\bar{F}'$ *with the strong topology is also a normal subspace of* E (in general, in the following, F will be a Hilbert space). We must naturally ensure that the identification of $\bar{F}'$ with a subspace of E is not in contradiction with the other identifications previously made. For example, if $F = \mathscr{H}$ is Hilbert, the identification of $\bar{\mathscr{H}}'$ with a subspace of E is incompatible with the identification θ of $\bar{\mathscr{H}}'$ with $\mathscr{H}$ coming from the Hilbert structure of $\mathscr{H}$. (There is one exceptional case: $\mathscr{H} = L^2$ if $I \geqq 0$. The inclusion of $\bar{\mathscr{H}}'$ in E by the first identification is indeed u^*, and the inclusion defined by the second is $j\theta$; they coincide if and only if $u^* j\theta$ or $u = \theta j^*$, that is to say, if $I = ju$ is the kernel $j\theta j^*$ of the Hilbert subspace $\mathscr{H}$ of E, or if $Ii \geqq 0$ and $\mathscr{H} = L^2$).

In the whole of this section, *the identification of $\bar{F}'$ with a subspace of E will be the only valid identification.* This identification can be expressed as follows: for $e' \in E'$ and $f' \in F'$:

$$(10.3) \qquad \begin{cases} \langle \bar{e}, f' \rangle_{F,F'} & = \overline{\langle \bar{f}', e' \rangle_{E,E'}}, \quad \text{or} \\ (\bar{e}' \mid \bar{f}')_{(\bar{E}',E) \text{ or } (E,\bar{E}')} & = (\bar{e}' \mid \bar{f}')_{(\bar{F}',F) \text{ or } (F,\bar{F}')} \end{cases} .$$

Or alternatively: we identify $\bar{f}' \in \bar{F}'$ with the element of E which defines on E' the weakly continuous linear form $e' \mapsto \overline{\langle f', \bar{e}' \rangle}_{F',F}$.

Or finally: we identify $\bar{F}'$ with the subspace of E consisting of the e such that the linear form $\bar{e}' \mapsto \overline{\langle e, e' \rangle_{E,E'}}$ is continuous on $\bar{E}'$ equipped with the topology induced by F. When there are contragredient anti-involutions on E and E' with $\bar{I} = I$, we will no longer speak of $\bar{E}$ and $\bar{E}'$, but only of E and E'; then F, $\bar{F}$, F' and $\bar{F}'$ will all be subspaces of E; if $F = \bar{F}$, we also have: $F' = \bar{F}'$. For example, if $E = \mathscr{D}'(X)$, the identification of F' with the subspace of $\mathscr{D}'$ is given by

$$(10.4) \qquad \langle T, \phi \rangle_{\mathscr{D}',\mathscr{D}} = \langle T, \phi \rangle_{F',F}, \qquad \phi \in \mathscr{D}, T \in F'.$$

($T \in F'$ is identified with the distribution $\phi \to \langle T, \phi\rangle_{F',F}$; $T \in \mathscr{D}'$ belongs to the subspace F' if and only if $\phi \to \langle T, \phi\rangle$ is continuous on $\mathscr{D}$ equipped with the topology induced by F).

And then $\bar{F}'$ is the conjugate of F' in $\mathscr{D}'$. We can define it directly by

$$(10.5) \qquad (T \mid \phi)_{\mathscr{D}',\mathscr{D}} = (T \mid \phi)_{\bar{F}',F}, \qquad \phi \in \mathscr{D}, T \in \bar{F}'.$$

($T \in \bar{F}'$ is identified with the distribution $\phi \mapsto \langle T, \phi\rangle_{\bar{F}',\bar{F}}$; $T \in \mathscr{D}'$ belongs to the subspace $\bar{F}'$ if and only if $\phi \mapsto \overline{\langle T, \bar{\phi}\rangle}$ is continuous on $\mathscr{D}$ equipped with the topology induced by F).

What we call here a normal subspace of E is what we have elsewhere called subspace of normal distributions on X[51].

Proposition 28b. *Let $\mathscr{H}$ be a normal subspace of E, with kernel H, and let $\bar{H}'$ be the kernel of $\bar{\mathscr{H}}'$ as a Hilbert subspace of E. Then H and $\bar{H}'$, linear maps from $\bar{E}'$ into E, extend in a unique manner to continuous linear maps $\hat{H}$ and $\hat{\bar{H}}'$ from $\bar{\mathscr{H}}'$ into $\mathscr{H}$ and from $\mathscr{H}$ into $\bar{\mathscr{H}}'$ respectively, which are inverses of each other and are inverse canonical isomorphisms between $\mathscr{H}$ and $\mathscr{H}'$.*

Proof. H is defined by (4.1) as being the composition $\bar{E}' \xrightarrow{j^*} \bar{\mathscr{H}}' \xrightarrow{\theta} \mathscr{H} \xrightarrow{j} E$, where j is the inclusion of $\mathscr{H}$ in E, j^* the inclusion of $\bar{E}'$ in $\bar{\mathscr{H}}'$ and θ the canonical isomorphism of $\bar{\mathscr{H}}'$ onto $\mathscr{H}$; so H extends to $\hat{H} = \theta$ and the extension is unique since $\bar{E}'$ is dense in $\bar{\mathscr{H}}'$. Likewise $\bar{H}'$ is defined by $\bar{E}' \to \mathscr{H} \xrightarrow{\theta^{-1}} \bar{\mathscr{H}}' \to E$, and extends to $\hat{\bar{H}}' = \theta^{-1}$, and the Proposition is shown. ☐

Thus we see the relationship between the kernels H and $\bar{H}'$ from $\mathscr{H}$ into $\bar{\mathscr{H}}'$ in E: *in some sense, they are inverses of each other.*

Construction of $\mathscr{H}$ and $\bar{\mathscr{H}}'$ by the kernel H of $\mathscr{H}$.

We know how to construct $\mathscr{H}$ from its kernel H: $\mathscr{H}$ is the completion in E of the space $\mathscr{H}_0 = H(\bar{E}')$ equipped with the norm $\|H\bar{e}'\|_{\mathscr{H}} = \langle H\bar{e}', e'\rangle^{1/2}$ (Proposition 10). But, if $\mathscr{H}$ is normal, we can also construct $\bar{\mathscr{H}}'$ from the kernel H of $\mathscr{H}$:

Proposition 29. *Let $\mathscr{H}$ be a normal subspace of E, with kernel H. Then $\bar{\mathscr{H}}'$ is the completion of $\bar{E}'$ in E, equipped with the norm*

$$(10.6) \qquad \|\bar{e}'\|_{\bar{\mathscr{H}}'} = \langle H\bar{e}', e'\rangle^{1/2} \qquad \textit{and with scalar product:}$$

$$(10.7) \qquad (\bar{e}' \mid \bar{f}')_{\bar{\mathscr{H}}'} = \langle H\bar{e}', f'\rangle.$$

Proof. $\bar{E}'$ is dense in $\bar{\mathscr{H}}'$, so $\bar{\mathscr{H}}'$ is the completion of $\bar{E}'$ in E equipped with the norm induced by $\bar{\mathscr{H}}'$. But since $\hat{H} = \theta$, a canonical isomorphism of $\bar{\mathscr{H}}'$ on $\mathscr{H}$ (Proposition 28b), we have $(\bar{e}' \mid \bar{f}')_{\bar{\mathscr{H}}'} = (H\bar{e}' \mid H\bar{f}')_{\mathscr{H}}$, and so we have (10.7) and then also (10.6). ☐

In the same vein, let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert subspaces of E, with $\mathscr{H}$ normal; and let v be a continuous linear map from E into E. We saw in Proposition 22b how to characterise inclusions $v(\bar{\mathscr{H}}') \leqq \mathscr{K}$ or $\geqq \mathscr{K}$, by the inequalities involving v and the kernels $\bar{H}'$ and K of $\bar{\mathscr{H}}'$ and $\mathscr{K}$. But it is interesting to

[51] See Schwartz [3], page 7.

have the inequalities involving v and the kernels H and K of $\mathscr{H}$ and $\mathscr{K}$. We will simply remark, by reusing (4.7), that

$$\text{(10.7b)} \qquad \begin{cases} \|Hv^*e'\|_{\bar{\mathscr{H}}'} = \|v^*\bar{e}'\|_{\mathscr{H}} = \\ \sup\limits_{f'\in E'} \frac{|\langle v^*\bar{e}', f'\rangle|}{\langle H\bar{f}', f'\rangle^{1/2}} = \sup\limits_{f'\in E'} \frac{|\langle v\bar{f}', e'\rangle|}{\langle H\bar{f}', f'\rangle^{1/2}}. \end{cases}$$ From this:

Proposition 29b. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert subspaces of E, with $\mathscr{H}$ normal; let H and K be their kernels. Let v be a continuous linear map from E into E.*

1°) *The necessary and sufficient condition for $v(\bar{\mathscr{H}}') \leqq \mathscr{K}$ is that for every $e', f' \in E'$, we have*

$$\text{(10.7c)} \qquad |\langle v\bar{f}', e'\rangle| \leqq \langle H\bar{f}', f'\rangle^{1/2}\langle K\bar{e}', e'\rangle^{1/2}.$$

This condition is satisfied if, for every $e' \in E'$, we have

$$\text{(10.7d)} \qquad |\langle v\bar{e}', e'\rangle| \leqq \frac{1}{2}\langle H\bar{e}', e'\rangle^{1/2}\langle K\bar{e}', e'\rangle^{1/2}.$$

2°) *The necessary and sufficient condition for $v(\bar{\mathscr{H}}') \geqq \mathscr{K}$ is that we have, for every $e' \in E'$:*

$$\text{(10.7e)} \qquad \sup_{f'\in E'} \frac{|\langle v\bar{f}', e'\rangle|}{\langle H\bar{f}', f'\rangle^{1/2}} \geqq \langle K\bar{e}', e'\rangle^{1/2}.$$

This condition is satisfied if we have, for every $e' \in E'$:

$$\text{(10.7f)} \qquad |\langle v\bar{e}', e'\rangle| \geqq \langle H\bar{e}', e'\rangle^{1/2}\langle K\bar{e}', e'\rangle^{1/2}.$$

As we remarked after Proposition 22b, the even stronger condition

$$\text{(10.7g)} \qquad \mathrm{Re}\langle v\bar{e}', e'\rangle \geqq \langle H\bar{e}', e'\rangle^{1/2}\langle K\bar{e}', e'\rangle^{1/2}$$

is known under the name of coercivity.

Let us take, for example, an open subset X of $\mathbb{R}^n$, and $\mathscr{H} = \mathscr{K} = \mathscr{H}^{-s}(X)$, defined on page 8. Its kernel is D, the differential operator in Equation (4.11).

Then $\bar{\mathscr{H}}' = \mathscr{H}_0^s(X)$ (see Example on page 70). The inequality (10.7f) is then, for every $\phi \in \mathscr{D}(X)$:

$$\left|\int_X (v\phi)\bar{\phi}dx\right| \geqq \|\phi\|^2_{\mathscr{H}_0^s}$$

which is sufficient to ensure $v(\mathscr{H}_0^s) \geqq \mathscr{H}^{-s}$; the inequality

$$\text{(10.7h)} \qquad \left|\int_X (v\phi)\bar{\phi}dx\right| \geqq \text{const.}\ \|\phi\|^2_{\mathscr{H}_0^s}$$

is sufficient to ensure $v(\mathscr{H}_0^s) \supset \mathscr{H}^{-s}$. We recover some well-known facts.

Proposition 30. *Let $\mathscr{H}$ be a Hilbert subspace of E, with kernel H. For $\bar{E}'$ to be a (not necessarily dense) subspace of $\mathscr{H}$ (with weakly continuous inclusion),* *it is necessary and sufficient that $\bar{E}'_{\bar{\mathscr{H}}'}$, with which we denote $\bar{E}'$ equipped with the seminorm (10.6), has a continuous inclusion in E, or that its unit ball*

$$B = \{\bar{e}' \in \bar{E}' : \langle H\bar{e}', e' \rangle \leqq 1\} \tag{10.8}$$

is bounded in E ((10.6) is then a norm).

For $\mathscr{H}$ to be normal (in other words, for $\bar{E}'$ to be additionally dense in $\mathscr{H}$), it is necessary and sufficient that $\bar{E}'_{\bar{\mathscr{H}}'}$ additionally has a completion in E (which is then $\bar{\mathscr{H}}'$), or that B is closed in $\bar{E}'$ with respect to the topology induced by E. (B is then the intersection of $\bar{E}'$ with the unit ball of $\bar{\mathscr{H}}'$).

Proof. 1°) To say $\bar{E}' \subset \mathscr{H}$ is to say (Proposition 8) that, for every $\bar{f}' \in \bar{E}'$, $\sup_{\langle H\bar{e}', e' \rangle \leqq 1} |\langle \bar{f}', e' \rangle|$ is finite; by replacing $|\langle \bar{f}', e' \rangle|$ by $|\langle f', \bar{e}' \rangle|$, this implies that B is weakly bounded in E, or strongly bounded, or that the inclusion $\bar{E}'_{\bar{\mathscr{H}}'} \to E$ is continuous.

2°) If $\mathscr{H}$ is normal, we saw in Proposition 29 that $\bar{E}'_{\bar{\mathscr{H}}'}$ has a completion in E, which is $\bar{\mathscr{H}}'$. Let us conversely suppose that $\bar{E}'_{\bar{\mathscr{H}}'}$ has a completion in E, say $\mathscr{K}$. Then $\mathscr{K}$ is normal. So $\bar{\mathscr{K}}'$ is also normal. But it is, by what was said after (10.3), the set of the h such that $\bar{e}' \to \overline{\langle h, e' \rangle}$ is continuous with respect to the topology of $\bar{E}'_{\bar{\mathscr{H}}'}$; in other words, the set of the h which satisfy (4.6), so it is $\mathscr{H}$, which is normal.

We know moreover that (Proposition 1) $\bar{E}'_H$ has a completion in E, if and only if B is closed in $\bar{E}'$, with respect to the topology induced by E.

□

Remark 1. Let us suppose that the kernel H of $\mathscr{H}$ extends to a weakly continuous linear map from E into itself. Then the inverse image under H of the unit ball of $\mathscr{H}$, a weakly compact and therefore weakly closed subset of E, is weakly closed in E; its intersection with $\bar{E}$, which is just B, is closed in $\bar{E}'$ with respect to the topology induced by E, So, in this case, if B is bounded in E, it is sufficient to ensure that $\mathscr{H}$ is normal. We can also see this in another way. $H = H^*$ maps $\bar{E}'$ into $\bar{E}'$; and $\mathscr{H}_0 = H(\bar{E}') \subset \bar{E}'$ is dense in $\mathscr{H}$; it is therefore sufficient that $\bar{E}'$ is contained in $\mathscr{H}$ for it to be dense there and for $\mathscr{H}$ to be normal.

Remark 2. Whether or not $\mathscr{H}$ is normal, the inclusion $j : \mathscr{H} \to E$ always has an adjoint $j^* : \bar{E}' \mapsto \bar{\mathscr{H}}'$ (but $\bar{\mathscr{H}}'$ is not necessarily identifiable with a subspace of E). We always have: $\|\bar{e}'\|_{\bar{\mathscr{H}}'}$ (defined by (10.6)) $= \langle H\bar{e}', e' \rangle^{1/2} = \|H\bar{e}'\|_{\mathscr{H}} = \|\theta j^*\bar{e}'\|_{\mathscr{H}} = \|j^*\bar{e}'\|_{\bar{\mathscr{H}}'}$ (norm of $j^*\bar{e}'$ in $\bar{\mathscr{H}}'$). If $\mathscr{H}$ is not dense in E, j^* is not injective, and $\|\bar{e}'\|_{\bar{\mathscr{H}}'}$ is only a seminorm. In any case, j^* is a dense isometry[52] from $\bar{E}'_{\bar{\mathscr{H}}'}$ into $\bar{\mathscr{H}}'$ and allows us to consider $\bar{\mathscr{H}}'$ as a complete Hausdorff space in $\bar{E}'_{\bar{\mathscr{H}}'}$; which justifies the notations $\bar{E}'_{\bar{\mathscr{H}}'}$ and $\|\bar{e}'\|_{\bar{\mathscr{H}}'}$. If $\mathscr{H}$ is normal, j^* is injective, and $\bar{\mathscr{H}}'$ is the completion of $\bar{E}'_{\bar{\mathscr{H}}'}$, its completion in E.

Corollary. *Let $\mathscr{H}_i$, $i \in I$ be a finite set of Hilbert spaces. Let u_i be a continuous linear map from $\mathscr{H}_i$ into E, and let $\mathscr{H} = \sum_{i \in I} u_i(\mathscr{H}_i)$, with kernel H.*

[52] This isometry is not injective if $\bar{E}'_{\bar{\mathscr{H}}'}$ is not Hausdorff.

Let us identify $\bar{\mathscr{H}}_i'$ with $\mathscr{H}_i$, and let $u_i^* : \bar{E}' \to \mathscr{H}_i$ be the adjoint of u_i. Let us equip $\bar{E}'$ with the scalar product

$$(\bar{e}' \mid \bar{f}')_{\bar{\mathscr{H}}'} = \sum_{i \in I} (u_i^* \bar{e}' \mid u_i^* \bar{f}')_{\mathscr{H}_i} \tag{10.9}$$

and with the norm

$$\|\bar{e}'\|^2_{\bar{\mathscr{H}}'} = \sum_{i \in I} \|u_i^* \bar{e}'\|^2_{\mathscr{H}_i}\,; \tag{10.10}$$

it becomes a pre-Hilbert space $\bar{E}'_{\bar{\mathscr{H}}'}$.

In order to have $\bar{E}' \subset \mathscr{H}$ (with weakly continuous inclusion), it is necessary and sufficient that the inclusion $\bar{E}'_{\bar{\mathscr{H}}'} \to E$ is continuous; for $\mathscr{H}$ to be normal, it is necessary and sufficient that $\bar{E}'_{\bar{\mathscr{H}}'}$ moreover has a completion in E, and it is then $\bar{\mathscr{H}}'$.

Proof. If we identify $\bar{\mathscr{H}}_i'$ with $\mathscr{H}_i$, the kernel of $\mathscr{H}_i$ in E is $u_i u_i^*$ (Proposition 21, as the kernel of $\mathscr{H}_i$ in itself is the identity), so the kernel of $\mathscr{H} = \sum_{i \in I} \mathscr{H}_i$ in E is $H = \sum_{i \in I} u_i u_i^*$. Then (10.9) and (10.10) are nothing but (10.7) and (10.6) with H, because:

$$\begin{aligned} \langle H\bar{e}', f' \rangle = \sum_{i \in I} \langle u_i u_i^* \bar{e}', f' \rangle_{E,E'} &= \sum_{i \in I} (u_i u_i^* \bar{e}' \mid \bar{f}')_{E,\bar{E}} = \\ &= \sum_{i \in I} (u_i^* \bar{e}' \mid u_i^* \bar{f}')_{\mathscr{H}}, \end{aligned} \tag{10.11}$$

and it then suffices to apply Proposition 30. □

Remark 1. Let us suppose that the $\mathscr{H}_i$ are in E, and that the u_i can be extended to weakly continuous linear maps $\bar{E}' \to \bar{E}'$, and $E \to E$. Then u_i^* can likewise be extended, and so $u_i u_i^*$ and H can also be extended. Then we can apply the remark which follows Proposition 30; it suffices that $\bar{E}' \subset \mathscr{H}$ for $\mathscr{H}$ to be normal.

Remark 2. Let us suppose that one of the $\mathscr{H}_i$ is a subspace of E containing $\bar{E}'$, with the corresponding map u_i being the inclusion of $\mathscr{H}_i$ in E; then a fortiori $\bar{E}' \subset \mathscr{H}$.

Example. Let us go back to Example 2 on page 8. We have $E = \mathscr{D}'(X)$, where X is an open subset of $\mathbb{R}^n$, $\mathscr{H}_p = L^2(X)$ for every index $p = (p_1, p_2, ..., p_n)$ of order $|p| \leqq s$, and $u_p = \sqrt{a_p} D^p$, the differentiation, a continuous linear map from $\mathscr{H}_p$ into E. Thanks to the natural anti-involution of $\mathscr{D}'$, we take $\bar{E}' = \mathscr{D}(X)$. Then $u_p^* = (-1)^{|p|} \sqrt{a_p} D_p$, a map from $\bar{E}'$ into $\mathscr{H}_p$. As the D^p can be extended to $\mathscr{D} \to \mathscr{D}$ and $\mathscr{D}' \to \mathscr{D}'$, and for $p = (0, 0, ..., 0)$, we have $\mathscr{H}_p = L^2$ with u_p being the inclusion $L^2 \to \mathscr{D}'$, and the preceding Remarks 1 and 2 show that $\mathscr{H}$ is normal. In fact we see directly that $\mathscr{D}$, equipped with the scalar product (10.9) which is nothing but (1.1), has a continuous inclusion in $\mathscr{D}'(X)$ and a completion in $\mathscr{D}'(X)$ which is nothing but $\mathscr{H}_0^s(X)$, the closure of $\mathscr{D}$ in $\mathscr{H}^s$. We thus have

$$\mathscr{H} = \sum_{|p| \leqq s} \sqrt{a_p} D^p L^2 = (\bar{\mathscr{H}}_0^s)' = \mathscr{H}^{-s}, \qquad \text{and} \qquad \bar{\mathscr{H}}' = \mathscr{H}_0^s.$$

The kernel of $\mathscr{H} = \mathscr{H}^{-s}$ in $\mathscr{D}'$ is thus the differential operator

$$D = \sum_{|p| \leqq s} (-1)^{|p|} a_p D^{2p}$$

(Equation (4.11), Example 2 on page 23, considered as an operator from $\mathscr{D}$ into $\mathscr{H}^{-s}$ or $\mathscr{D}'$). According to Proposition 28b, D is an isomorphism from $\bar{\mathscr{H}}' = \mathscr{H}_0^s$ onto $\mathscr{H}^{-s}$; it is a classical theorem from the theory of elliptic partial differential equations. The inverse isomorphism $G : \mathscr{H}^{-s} \to \mathscr{H}_0^s$ is called the Green's operator of D; it is the kernel of the Hilbert subspace $\mathscr{H}_0^s$ of $\mathscr{D}'$ (Proposition 28b), if we consider it as an operator $\mathscr{D} \to \mathscr{H}_0^s \to \mathscr{D}'$. D and G are the canonical inverse isomorphisms between $\mathscr{H} = \mathscr{H}^{-s}$ and $\bar{\mathscr{H}}' = \mathscr{H}_0^s$.

Remark. Let us now suppose that some of the a_p are zero. Remark 2 is no longer applicable, because none of the operators u_p are now the identity. And indeed, it is no longer necessarily correct that $\mathscr{H}$ is normal; this depends on the a_p, X and n. It is always normal if X is bounded in $\mathbb{R}^n$ and at least one of the a_p is non-zero, because we know then that $\mathscr{D}_H$ still has a continuous inclusion in $\mathscr{D}'$[53], and consequently has a completion in $\mathscr{D}'$ (Remark 1). The kernel of $\mathscr{H}$ is always the differential operator $\sum_{|p|=s} (-1)^{|p|} a_p D^{2p}$; if $\mathscr{H}$ is normal, this operator has a Green's operator, which is the kernel of $\bar{\mathscr{H}}'$ in $\mathscr{D}'$.

Case of extendable kernels.

We will say that a kernel $v : \bar{E}' \to E$ is extendable, if it is weakly continuous from $\bar{E}'$ into $\bar{E}'$, and extends (necessarily in a unique manner, since $\bar{E}'$ is dense in E) to a weakly continuous linear map from E into E, which we again denote by v. Its adjoint v^* is then also extendable. We have seen such extendable kernels in the preceding remarks and examples. For $E = \mathscr{D}'(X)$ and $\bar{E}' = \mathscr{D}(X)$, the differential operators with C^∞ coefficients are extendable; for $X = \mathbb{R}^n$, the convolution with a distribution with compact support is extendable (the kernel of an extendable map in the theory of kernels is also called "compact regular"). For $E = \mathscr{S}'$ and $\bar{E}' = \mathscr{S}$, the Fourier transformation is extendable, it is even an isomorphism. If v is Hermitian, i.e. $v^* = v$, it suffices that it is weakly continuous $\bar{E}' \to \bar{E}'$, or extends to a weakly continuous map $E \to E$, for it to have both properties together, and so for it to be extendable.

Let v be an extendable kernel, and w any kernel. Then we can compose v and w, and consider vw: $\bar{E}' \overset{w}{\to} E \overset{v}{\to} E$, and of wv: $\bar{E}' \overset{v}{\to} \bar{E}' \overset{w}{\to} E$, as new kernels. More generally, we can compose, in any order, several kernels, if all of them, except at most one, are extendable; this law of composition is associative.

If v is extendable, we say that w is a left-inverse of v (resp. right-inverse) if $wv = I$ (resp. $vw = I$), where I is the inclusion of $\bar{E}'$ in E, extendable to the identity $\bar{E}' \to \bar{E}'$ and $E \to E$. If w is a left- and right-inverse of v, it is said to be a bilateral inverse, or simply an inverse. If w and v are Hermitian and if w is a left- or right-inverse of v, it is a bilateral inverse. Let us note that $\mathscr{L}(\bar{E}'; E)$ is not an algebra, and that a kernel v can have infinitely many inverses. If it

(53) More generally, if $P(D)$ is a differential operator with constant coefficients, we have, for every bounded open subset X of $\mathbb{R}^n$, an inequality $\|\phi\|_{L^2} \leqq \text{const.}\|P(D)\phi\|_{L^2}$; this type of inequalities, well known for a long time for some particular operators, has been introduced in general by Malgrange [1], and generalised by Hörmander [1].

has an extendable inverse, it is a weak isomorphism from $\bar{E}'$ onto $\bar{E}'$ and from E onto E, and then it is its only inverse. If, for example, $E = \mathscr{D}'$ and $\bar{E}' = \mathscr{D}$, and if v is a C^∞ differential operator, a (left- or right-) inverse is also called a (left- or right-) elementary kernel of v. It is well known that, if it exists, there are in general infinitely many of them.

Proposition 31. *Let $\mathscr{H}$ be a Hilbert subspace of E, with extendable kernel H. For $\mathscr{H}$ to be normal, it is necessary and sufficient that H has an inverse $L \geqq 0$; in this case, there are in general infinitely many inverses, and the kernel $\bar{H}'$ of $\bar{\mathscr{H}}'$ is the smallest non-negative kernel that is an inverse of H.*

We have $L = \bar{H}' + N$, where $\bar{H}'$ and N are alien; and we have $HN = NH = 0$. The Hilbert subspace $\mathscr{L}$ with kernel L is a direct sum of two closed orthogonal subspaces, $\mathscr{N} = \mathscr{L} \cap H^{-1}(\{0\})$, and $\bar{\mathscr{H}}'$, the closure of $\bar{E}'$ in $\mathscr{L}$. The operator $H\bar{H}'$ is the identity on $\mathscr{H}$, and the operator $\bar{H}'H$ is the orthogonal projection of $\mathscr{L}$ onto $\bar{\mathscr{H}}'$ (so the identity on $\bar{\mathscr{H}}'$).

Proof. 1°) Let us suppose that $\mathscr{H}$ is normal. The extension $\hat{H} : \bar{\mathscr{H}}' \to \mathscr{H}$ of H defined in Proposition 22b is nothing but H itself, assumed to be extended to all of E (indeed, H and $\hat{H}$ are both weakly continuous from $\bar{\mathscr{H}}'$ into E and coincide on $\bar{E}'$, which is dense in $\bar{\mathscr{H}}'$). Then we have seen, in Proposition 22b, that the kernel $\bar{H}'$ of $\bar{\mathscr{H}}'$ extends to $\hat{\bar{H}} : \mathscr{H} \to \bar{\mathscr{H}}'$, and that $\hat{H}$ and $\hat{\bar{H}}'$ are inverses of each other: this implies that $\bar{H}'$ is an inverse of H in the sense given just before Proposition 31.

2°) Conversely, let us suppose that H has an inverse $L \geqq 0$. Let $\mathscr{L}$ be the Hilbert subspace of E of kernel L. The image $H(\mathscr{L})$ of $\mathscr{L}$ under H has $HLH^* = HLH = H$ as its kernel, by Proposition 21 applied to the weakly continuous linear operator H from E into E, since $HL = I$ (or $LH = I$). So this image is the space $\mathscr{H}$. As then $\bar{E}' = HL(\bar{E}')$, and $L(\bar{E}') \subset \mathscr{L}$, we have $\bar{E}' \subset H(\mathscr{L}) = \mathscr{H}$, and by Remark 1 after Proposition 30, $\mathscr{H}$ is indeed normal. [Let us remark that we see directly that $\bar{E}'$ is dense in $\mathscr{H}$: because $L(\bar{E}')$ is dense in $\mathscr{L}$ (Proposition 7), and H is continuous and surjective from $\mathscr{L}$ onto $\mathscr{H}$ since $H(\mathscr{L}) = \mathscr{H}$.] Proposition 21 moreover says this: if $\mathscr{N} = (H^{-1}\{0\}) \cap \mathscr{L}$, and if $\mathscr{K}$ is the orthogonal complement of $\mathscr{N}$ in $\mathscr{L}$, then $HL(\bar{E}') = \bar{E}'$ is a dense subspace of $\mathscr{K}$. $\bar{E}'$ is also a dense subspace of $\bar{\mathscr{H}}'$; we are going to see that, on $\bar{E}'$, the norms induced by $\mathscr{K}$ and $\bar{\mathscr{H}}'$ coincide. Indeed, by (4.4) applied to L and to $H\bar{e}' \in \bar{E}'$:

$$\begin{aligned}(10.12)\qquad \|\bar{e}'\|_{\mathscr{K}} = \|\bar{e}'\|_{\mathscr{L}} = \|LH\bar{e}'\|_{\mathscr{L}} &= \left\langle LH\bar{e}', \overline{H\bar{e}'}\right\rangle^{1/2}_{E,E'} \\ = \left\langle HLH\bar{e}', e'\right\rangle^{1/2}_{E,E'} &= \left\langle H\bar{e}', e'\right\rangle^{1/2} = \|\bar{e}'\|_{\bar{\mathscr{H}}'}\end{aligned}$$

by (10.6).

Then $\mathscr{K}$ and $\bar{\mathscr{H}}'$ coincide, as completions in E of $\bar{E}'$ with respect to the norm (10.12). As $\mathscr{K}$ is a subspace of $\mathscr{L}$ with respect to the induced norm, we have $\bar{\mathscr{H}}' = \mathscr{K} \leqq \mathscr{L}$, so $H' \leqq L$: $\bar{H}'$ is the smallest non-negative kernel that is an inverse of H. The rest of the statement is evident; $H\bar{H}'$ is the identity on $\mathscr{H}$ by Proposition 22b; $\bar{H}'H$ is defined on $\mathscr{L}$, and since $H(\mathscr{L}) = \mathscr{H}$ and it is the identity on $\bar{\mathscr{H}}'$ and 0 on $\mathscr{N}$, it is thus the orthogonal projection of $\mathscr{L}$ onto $\bar{\mathscr{H}}'$.

□

Remark 1. We knew before that $\bar{\mathscr{H}}' \cap H^{-1}(\{0\}) = \{0\}$, since H is an isomorphism from $\bar{\mathscr{H}}'$ onto $\mathscr{H}$. Proposition 31 gives the general form of kernels $L \geqq 0$ that are inverses of H: $L = \bar{H}' + N$, with $N \geqq 0$, $HN = NH = 0$, and the general structure of corresponding Hilbert subspaces: $\mathscr{L} = \bar{\mathscr{H}}' + \mathscr{N}$, with $\mathscr{N}$ an arbitrary Hilbert subspace of $H^{-1}(\{0\}) \subset E$.

Remark 2. We must be careful of these "inverses" of H and remember that $\mathscr{L}(\bar{E}'; E)$ is not an algebra. $\bar{H}'$ is an inverse of H, and $\bar{H}'H$ is equal to the identity on $\mathscr{D}$ and on $\bar{\mathscr{H}}'$, but not on $\mathscr{L}$! In the same train of thought, we could believe that the kernels L and N, like the kernel $\bar{H}'$, extend to continuous operators from $\mathscr{H}$ into $\mathscr{L}$. *This is not true: if $\mathscr{N} \neq \{0\}$, L and N are never continuous from $\bar{E}'$ equipped with the topology induced by $\mathscr{H}$ in E.*

If indeed one of the two were continuous, the other would also be continuous, since $L - N = \bar{H}'$ is continuous, from $\mathscr{H}$ onto $\bar{\mathscr{H}}'$. And then L and N would extend, by continuity, to continuous linear maps from $\mathscr{H}$ into E, which is quasi-complete. But NH is zero on $\bar{E}'$, so N is zero on $H(\bar{E}')$, and dense in $\mathscr{H}$ (Proposition 7); so N would be zero on $\mathscr{H}$. But then $N(\bar{E}') \subset N(\mathscr{H})$ would also be $\{0\}$, and so it has to be dense in $\mathscr{N}$.

Application to limit problems of Neumann type

We frequently pose problems of elliptic limits in the following way. We start with a Hilbert subspace $\mathscr{L}$ of $E = \mathscr{D}'(X)$, where X is an open subset of $\mathbb{R}^n$. We will assume that $\bar{E}'$ is contained in $\mathscr{L}$, with weakly continuous inclusion, but not necessarily dense (and in fact, the most interesting cases are those where it is not dense). For every l in $\mathscr{L}$, the linear map $e' \mapsto (l \mid \bar{e}')$ is weakly continuous on E', so defines an element Hl of $\mathscr{L}$ with

$$\langle Hl, e' \rangle = (l \mid \bar{e}')_{\mathscr{L}}. \tag{10.13}$$

H is a weakly continuous, hence continuous, linear map from $\mathscr{L}$ into E.

(This operator is, if we identify $\mathscr{L}$ with its anti-dual $\bar{\mathscr{L}}'$, the adjoint of the inclusion of $\bar{E}'$ in $\mathscr{L}$). We assume that H is a C^∞ elliptic differential operator, so an extendable operator in the sense defined on page 71. Then $\mathscr{L}$ defines a limit problem relative to the elliptic operator H, as follows. We say that an element l of $\mathscr{L}$ satisfies Neumann's homogeneity condition relative to $\mathscr{L}$ if Hl is in $\bar{E}'$, and if we have

$$\langle Hl, \bar{k} \rangle = (l \mid k)_{\mathscr{L}} \qquad \text{for every } k \text{ in } \mathscr{L}. \tag{10.14}$$

Then let $\bar{e}' \in \bar{E}'$. We say that an element l in $\mathscr{L}$ satisfying the homogeneity condition (10.14) and

$$Hl = \bar{e}' \tag{10.15}$$

is a solution of Neumann's problem. It is simply equivalent to saying that we have, for $k \in \mathscr{L}$:

$$\langle \bar{e}', \bar{k} \rangle = (l \mid k)_{\mathscr{L}}, \qquad \text{or} \qquad \langle k, e' \rangle = (k \mid l)_{\mathscr{L}}. \tag{10.16}$$

It is not difficult to show the existence and uniqueness of such an element l, which we will denote by $L\bar{e}'$; L is called the Neumann kernel of Neumann's problem defined by $\mathscr{L}$. (10.16) becomes

$$\langle k, e' \rangle = (k \mid L\bar{e}')_{\mathscr{L}} \qquad \text{for every } k \in \mathscr{L}. \tag{10.17}$$

Let us interpret these results in terms of Hilbert subspace kernels. H is weakly continuous from $\bar{E}'$ into E, and trivially Hermitian and non-negative, because (10.13) gives

$$\langle H\bar{e}', e' \rangle = \|e'\|^2_{\mathscr{L}}. \tag{10.18}$$

So it defines a Hilbert subspace $\mathscr{H}$ of E; and by hypothesis H is extendable, so we can apply the previous results. On $\bar{E}'$, the norm $\bar{E}'_{\bar{\mathscr{H}}'}$ defined in Equation (10.6) is, by (10.17), nothing but the norm induced by $\mathscr{L}$. So the conditions of Proposition 30 are satisfied, $\mathscr{H}$ is normal, and the closure of $\bar{E}'$ in $\mathscr{L}$, which is the completion of $\bar{E}'_{\bar{\mathscr{H}}'}$ in E, is the anti-dual $\bar{\mathscr{H}}'$ of $\mathscr{H}$. On $\bar{\mathscr{H}}'$, H is the canonical isomorphism $\bar{\mathscr{H}}' \to \mathscr{H}$; its inverse $\bar{H}'$, the kernel of $\bar{\mathscr{H}}'$, is also called the Green's kernel of H. Let $\mathscr{N}$ be the orthogonal complement of $\bar{\mathscr{H}}'$ in $\mathscr{L}$; on $\mathscr{N}$, H vanishes by (10.13); so the space $\mathscr{L} = \bar{\mathscr{H}}' + \mathscr{N}$ has exactly the form indicated in Proposition 31, and its kernel is an inverse of H. This kernel is nothing but L, by the equality (10.17). We see thus that in reality, *as soon as $\mathscr{L}$ is given, the Neumann kernel L is immediately known*, even before any interpretation as a limit problem; it is the kernel of $\mathscr{L}$. The property $L - \bar{H}' = N \geqq 0$ results from Proposition 31. Dirichlet's problem is a special case of Neumann's problem, corresponding to $\mathscr{L} = \bar{\mathscr{H}}'$.

For example, if D is the differential operator (4.11), $\mathscr{H} = \mathscr{H}^{-s}$ and $\bar{\mathscr{H}}' = \mathscr{H}^s_0$ (Dirichlet's problem); Neumann's problem (corresponding to the cancellation of the normal derivatives on the boundary), corresponds to $\mathscr{L} = \mathscr{H}^s$. We thus retrieve what was stated on page 24.

Often, $\mathscr{L}$ is given as a Hilbert space, but we additionally give ourselves a continuous Hermitian bilinear form B on $\mathscr{L}$, which we take to be coercive, i.e. such that

$$B(l, l) \geqq \text{const.}\|l\|^2_{\mathscr{L}}. \tag{10.19}$$

And it is with respect to B and not $(\cdot \mid \cdot)_{\mathscr{L}}$ that we solve the problem. But we remark that the initial Hilbert structure of $\mathscr{L}$ is uninteresting, and that B defines an equivalent Hilbert structure, to which we apply the above.

Finally, if B is only a continuous sesquilinear form, not Hermitian but coercive, i.e. satisfying

$$\mathrm{Re}B(l, l) \geqq \text{const.}\,\|l\|^2_{\mathscr{L}}, \tag{10.20}$$

we can solve the same Neumann's problem relative to B. This time, the Neumann kernel is the kernel of $\mathscr{L}$, equipped with a "sesquilinear" structure (not Hilbert, nor even Hermitian in the sense of §12), that is to say, the structure of the form B. In this simple case, the kernel is defined as for a Hilbert structure. We define a continuous linear map γ from $\mathscr{L}$ into $\bar{\mathscr{L}}'$;

$$B(k \mid k) = (h \mid \gamma k)_{\mathscr{L}, \bar{\mathscr{L}}'}. \tag{10.21}$$

The coercivity formula shows that γ is a monomorphism, because $\|\gamma l\| \geqq \text{const.}\|l\|_{\mathscr{L}}$, but also that its adjoint, which again operates from $\mathscr{L}$ into $\bar{\mathscr{L}}'$, is a monomorphism, so it is an isomorphism from $\mathscr{L}$ onto $\bar{\mathscr{L}}'$; if θ is the inverse isomorphism $\bar{\mathscr{L}}' \to \mathscr{L}$, we define the kernel associated to $(\mathscr{L}; B)$ by $L = j\theta j^*$, and it is the Neumann kernel satisfying (10.17).

Finally, we will in general consider extending L to spaces that are larger than $\bar{E}'$. If F is a subspace of E, with weakly continuous normal inclusion, and if $\mathscr{L} \subset F$ with a weakly continuous inclusion, we can confirm that L extends to a weakly continuous linear map from $\bar{F}'$ into $\mathscr{L}$. For example, in the case cited above, where $\mathscr{L} = \mathscr{H}^s$, we can, if the boundary $\dot{X}$ of X is regular enough, take $F = \mathscr{H}^{-1/2}$[54], and L operates continuously from $\mathscr{H}^{-1/2}$ into $\mathscr{H}^s$. On the other hand, as we have indicated in Remark 2 after Proposition 31, for bounded X, L never operates from $\mathscr{H}^{-s}$ into $\mathscr{H}^s$.

§11. Applications in potential theory

Charges and potentials. Let X be an open subset of $\mathbb{R}^n$ and $D \geqq 0$ a kernel: $\mathscr{D}(X) \to \mathscr{D}'(X)$; so we have, $\forall \phi \in \mathscr{D}$: $\langle D\phi, \bar{\phi}\rangle \geqq 0$. D defines a Hilbert subspace $\mathscr{W}$ of $\mathscr{D}'(X)$; *importantly, in what follows, we assume that $\mathscr{W}$ is normal.* We call it the space of charges of finite energy[55]; it is the completion in $\mathscr{D}'$ of $D\mathscr{D}$, equipped with the norm

$$\|D\phi\|_{\mathscr{W}} = \langle D\phi, \bar{\phi}\rangle^{1/2}. \tag{11.1}$$

The square of the norm of a charge $T \in \mathscr{W}$ is called its *energy*. *$\bar{\mathscr{W}}' = \mathscr{U}$ is called the space of potentials of finite energy*; it is also normal; its kernel G is called *the Green's operator of D in X*. We saw (Proposition 22b) that D and G extend into the following operators, also denoted as D and G: $\mathscr{U} \xrightarrow{D} \mathscr{W}$ and $\mathscr{W} \xrightarrow{G} \mathscr{U}$, which are canonical isomorphisms between $\mathscr{W}$ and $\mathscr{U} = \bar{\mathscr{W}}'$ that are inverses of each other; the extended D is also called the *charge operator* and the extended G the *potential operator*; for a charge $T \in \mathscr{W}$, GT is its potential; for a potential $U \in \mathscr{U}$, DU is its charge.

We have, for $S \in \mathscr{W}$, $T \in \mathscr{W}$, $U \in \mathscr{U}$ and $V \in \mathscr{U}$,

$$\begin{cases} (S \mid T)_{\mathscr{W}} = \langle GS, \bar{T}\rangle_{\bar{\mathscr{W}}', \bar{\mathscr{W}}} = (GS \mid T)_{\bar{\mathscr{W}}', \mathscr{W}} \\ (U \mid V)_{\mathscr{U}} = \langle DU, \bar{V}\rangle_{\mathscr{W}, \mathscr{W}'} = (DU \mid V)_{\mathscr{W}, \bar{\mathscr{W}}'} \\ (T \mid DU)_{\mathscr{W}} = (T \mid U)_{\mathscr{W}, \bar{\mathscr{W}}'} = \langle T, \bar{U}\rangle_{\mathscr{W}, \mathscr{W}'} = (GT \mid U)_{\mathscr{U}}. \end{cases} \tag{11.2}$$

In potential theory, it is fairly common to start with D, then to find G, and then to construct $\mathscr{W}$ from G as the completion in $\mathscr{D}'$ (Proposition 29) of $\mathscr{D}$ equipped with the norm

$$\|\phi\|_{\mathscr{W}} = \langle G\phi, \bar{\phi}\rangle^{1/2}. \tag{11.3}$$

It is quite an illogical bidualisation, since $\mathscr{W}$ can be constructed by (11.1) without knowing G, even without the existence of $\mathscr{U}$ and G, that is to say, without $\mathscr{W}$ being normal. At the same time the relationship $\mathscr{U} = \bar{\mathscr{W}}'$ goes unnoticed in general.

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[54] If X is a bounded open subset of $\mathbb{R}^n$ with a regular enough boundary, $\mathscr{H}^s$ is normal if and only if $s \leqq 1/2$. See Lions [1], Chapter II.

[55] For the notions of charges and potentials of finite energy, and their principal properties, see, for example, Deny [1].

Balayage. *The balayage of $T \in \mathscr{W}$ on a closed subset F of $\mathbb{R}^n$ is the orthogonal projection (in $\mathscr{W}$) T_F^D of T onto the closed subspace $\mathscr{W}_F$ of $\mathscr{W}$ consisting of distributions with support in F.*

To have an interesting theory, we are in general obliged to make the following supplementary hypothesis on the anti-duality between $\mathscr{U}$ and $\mathscr{W}$:

(Supp) *If $U \in \mathscr{U}$ and $T \in \mathscr{W}$ have supports without common points, and if the support of T is compact, then $(U, T)_{\bar{\mathscr{W}}', \mathscr{W}} = \langle U, \bar{T}\rangle_{\bar{\mathscr{W}}', \bar{\mathscr{W}}} = 0$.*

As $\langle U, \bar{T}\rangle_{\bar{\mathscr{W}}', \bar{\mathscr{W}}}$ is always defined by passing to the limit of $\langle U, \bar{\phi}_v\rangle_{\mathscr{D}', \mathscr{D}}$ for the ϕ_v in $\mathscr{D}$ converging to T in $\mathscr{W}$, this property will of course be a consequence of the following:

Every $T \in \mathscr{W}$ with compact support is the limit in $\mathscr{W}$ of a sequence of functions in $\mathscr{D}$, with support in an arbitrary neighbourhood of the support of T.

We see that this property will in turn be a consequence of the *approximation property by regularisation*[(56)], often satisfied by $\mathscr{W}$.

Proposition 32. *The potentials of T and its balayage T_F^D are equal in the interior $\mathring{F}$ of F. If the hypothesis (Supp) is satisfied, two charges whose potentials coincide in the neighbourhood of a compact set F have the same balayage on F.*

Proof. The balayage T_F^D, the orthogonal projection of T onto $\mathscr{W}_F$, is *characterised* by $T_F^D \in \mathscr{W}_F$ and

$$\forall R \in \mathscr{W}_F, \qquad (T \mid R)_{\mathscr{W}} = (T_F^D \mid R)_{\mathscr{W}}. \tag{11.4}$$

By setting $R = \phi \in \mathscr{D}$ with support in $\mathring{F}$, and by using the first formula in (11.2), we see that GT and GT_F^D coincide on such functions ϕ so coincide in $\mathring{F}$. We see that we even have more: GT and GT_F^D have the same value on every $\phi \in \mathscr{D}$ with support in F.

If, moreover, F is compact and (Supp) is satisfied, and if $GT = 0$ in the neighbourhood of F, GT and R have disjoint supports for $R \in \mathscr{W}_F$, so $(GT \mid R)_{\bar{\mathscr{W}}', \bar{\mathscr{W}}} = 0$, so $(T \mid R)_{\mathscr{W}} = 0$, and the balayage of T on F is zero; by difference, we deduce the second part of the statement. ☐

If F is compact, there exists $T \in \mathscr{W}$ whose potential GT is equal to 1 in the neighbourhood of F; it suffices to take $\phi \in \mathscr{D}$ equal to 1 in the neighbourhood of F, and then to see that $\phi \in \mathscr{U}$ and $T = D\phi$ give what we want. If (Supp) is satisfied, the balayage T_F^D of T on F is independent of the choice of T, *it is the equilibrium distribution of F.* Its potential is 1 in the interior $\mathring{F}$ of F.

Restriction to an open subset $Y \subset X$

Now let Y be an open subset of X. There exists a natural inclusion $i : \mathscr{D}(Y) \to \mathscr{D}(X)$; for $\phi \in \mathscr{D}(Y)$, $i\phi$, also denoted ϕ, can be obtained by extending ϕ by 0 in $X \cap Y^c$. The transpose ${}^t i$ and the adjoint i^* are the operation of restriction $\rho : \mathscr{D}'(Y) \to \mathscr{D}'(X)$: for $T \in \mathscr{D}'(X)$, ρT, also denoted as T, is its restriction to the open subset Y of X. The kernel D also has a restriction ρD, again denoted as D, to Y: for $\phi \in \mathscr{D}(Y)$, $(\rho D)\phi = \rho(D\phi) = \rho D i\phi$ or $i^* D i\phi$:

$$\mathscr{D}(Y) \xrightarrow{i} \mathscr{D}(X) \xrightarrow{D} \mathscr{D}'(X) \xrightarrow{i^*} \mathscr{D}'(Y).$$

(56) Schwartz [3], page 8.

Of course, ρD is again a non-negative kernel on Y, and defines a Hilbert subspace of $\mathscr{D}'(Y)$ which is, by Proposition 21, nothing but the image $i^*\mathscr{W} = \rho\mathscr{W}$ of $\mathscr{W}$ under the restriction ρ (but, as defined by ρD, it only depends on the restriction of D to Y and not on the open set X on which D was initially defined; we also denote by $\mathscr{W}(Y)$ the space of charges of finite energy defined by D on Y). A distribution S of $\mathscr{D}'(Y)$ belongs to $\mathscr{W}(Y)$ if and only if it is the restriction ρT to Y of a distribution T of $\mathscr{W}$, and moreover

$$\|S\|_{\mathscr{W}(Y)} = \inf_{\rho T = S} \|T\|_{\mathscr{W}}\,. \tag{11.5}$$

If we go back to Proposition 21, $\mathscr{W}$ is the direct sum of two orthogonal spaces: $\mathscr{W}_F = \mathscr{W} \cap \rho^{-1}(\{0\})$, the set of charges with support in $F = X \cap Y^c$, and $\mathscr{W}_F^{\perp} = \mathscr{L}$, its orthogonal complement, the closure of $D\mathscr{D}(Y)$ in $\mathscr{W}$, the space of charges whose balayage on F vanishes. Then ρ is an isometry from $\mathscr{L}$ onto $\mathscr{W}(Y)$, so that the infimum in (11.5) is a minimum: for $S \in \mathscr{W}(Y)$, there exists a unique T in $\mathscr{W}$ such that $\rho T = S$ and $\|T\|_{\mathscr{W}} = \|S\|_{\mathscr{W}(Y)}$, and it is characterised by the fact that it is in $\mathscr{L}$. If $T \in \mathscr{W}$ is such that $\rho T = S$, and if T_F^D is its balayage on F, $T - T_F^D$ is the orthogonal projection of T on $\mathscr{L}$, and we have

$$\|S\|^2_{\mathscr{W}(Y)} = \|T - T_F^D\|^2_{\mathscr{W}} = \|T\| - \|T_F^D\|^2_{\mathscr{W}}. \tag{11.6}$$

Since $D\mathscr{D}(Y) \subset \mathscr{L}$, we have, for $\phi \in \mathscr{D}(Y)$,

$$\|D\phi\|_{\mathscr{W}(Y)} = \|D\phi\|_{\mathscr{W}}. \tag{11.6b}$$

The fact that $\mathscr{W}$ is normal in $\mathscr{D}'(X)$ does not necessarily imply that $\mathscr{W}(Y)$ is normal in $\mathscr{D}'(Y)$. If $\phi \in \mathscr{D}(Y)$, $i\phi$ is in $\mathscr{D}(X) \subset \mathscr{D}'(X)$, and its restriction $i^*i\phi$ is ϕ itself; as $\mathscr{W}$ is normal, $i\phi \in \mathscr{W}$, so $\phi = i^*i\phi \in \mathscr{W}(Y)$. Thus $\mathscr{D}(Y)$ is a subspace of $\mathscr{W}(Y)$, *but not necessarily dense*. The operator ρD thus does not necessarily admit a theory of potentials in Y. $\mathscr{D}(Y)$ is dense in $\mathscr{W}(Y)$ if and only if $i\mathscr{D}(Y) \subset \mathscr{W}$ has an orthogonal projection on $\mathscr{L}$ that is dense in $\mathscr{L}$.

Case where D is a C^∞ differential operator.

Henceforth, we restrict ourselves to this case (D is then extendable in $\mathscr{D}'(X)$ in the sense of the previous section on page 71). Then ρD is the same differential operator restricted to Y, we denote it also by D and it is again extendable in $\mathscr{D}'(Y)$. So (by Remark 1 after Proposition 30) $\mathscr{W}(Y)$ is normal: $D(\mathscr{D}(Y)) \subset \mathscr{D}(Y)$ is dense in $\mathscr{W}(Y)$ and D defines a theory of potentials in Y. We can also see this by applying Proposition 31: ρG is a non-negative inverse of ρD (for $\phi \in \mathscr{D}(Y)$, $\rho G \cdot \rho D \cdot \phi = \rho \cdot GD\phi = \phi$, $\rho D \cdot \rho G \cdot \phi = \rho \cdot DG\phi = \phi$), so $\mathscr{W}(Y)$ is normal. Moreover, Proposition 31 also shows that $\rho\mathscr{U}$ is a direct sum of two orthogonal closed subspaces: $\mathscr{N} = \rho\mathscr{U} \cap D^{-1}(\{0\})$, the space of $U \in \rho\mathscr{U}$ satisfying $DU = 0$ in Y, and $\mathscr{U}(Y) = \overline{\mathscr{W}(Y)'}$, which is the space of potentials of finite energy in Y, relative to the kernel D (and which only depends on the restriction of D to Y, and not on the open set X in which D was initially defined, nor on $\mathscr{W}$ or on $\mathscr{U}$ relative to X). This decomposition $\rho\mathscr{U} = \mathscr{N} + \mathscr{U}(Y)$ is crucial: the space of potentials of D in Y is not the restriction to Y of the space of potentials of D in X, it is smaller. If N is the kernel of $\mathscr{N}$ relative to $\mathscr{D}'(Y)$, and $G(Y)$ the kernel of $\mathscr{U}(Y)$, $G(Y)$ is the Green's operator of D for the open set Y; it is not the restriction $\rho G = G$ of G to Y, but we have

$\rho G = N + G(Y)$. $G(Y)$ is the smallest non-negative inverse of D in Y. (11.6) gives (since $G : \mathscr{W} \to \mathscr{U}$ and $G(Y) : \mathscr{W}(Y) \to \mathscr{U}(Y)$ are isometries), for $\phi \in \mathscr{D}(Y)$:

$$(11.7) \qquad \begin{cases} \langle G\phi, \bar{\phi}\rangle = \|G\phi\|^2_{\mathscr{U}} = \|\phi\|^2_{\mathscr{W}} \\ \langle G(Y)\phi, \bar{\phi}\rangle = \|G(Y)\phi\|^2_{\mathscr{U}(Y)} = \|\phi\|^2_{\mathscr{W}(Y)} = \|\phi\|^2_{\mathscr{W}} - \|\phi^D_F\|^2_{\mathscr{W}}. \end{cases}$$

By Proposition 31, we have $DN = ND = 0$; $D \cdot G(Y)$ is the identity on $\mathscr{W}(Y)$; $G(Y) \cdot D$ is the orthogonal projection of $\rho\mathscr{U}$ on $\mathscr{U}(Y)$.

Since G is an isometry from $\mathscr{W}$ onto $\mathscr{U}$, the orthogonal decomposition $\mathscr{W} = \mathscr{W}_F + \mathscr{L}$ gives an orthogonal decomposition $\mathscr{U} = G\mathscr{W}_F + G\mathscr{L}$. $G\mathscr{W}_F$ is the space of potentials U on X, satisfying $DU = 0$ in Y. In turn, we can decompose this into an orthogonal sum $\mathscr{U}_1 + \mathscr{U}_2$, where $\mathscr{U}_1 = \mathscr{U} \cap \rho^{-1}(\{0\})$ is the set $\mathscr{U}_F$ of the potentials on X with support in F, or zeros in Y, and $\mathscr{U}_2$ is its orthogonal complement in $G\mathscr{W}_F$; ρ is an isometry from $\mathscr{U}_2$ onto $\mathscr{N}$. Then $\mathscr{U} = \mathscr{U}_1 + \mathscr{U}_2 + \mathscr{U}_3$, where $\mathscr{U}_3 = G\mathscr{L}$; since $\mathscr{L}$ is the closure of $D(\mathscr{D}(Y))$ in $\mathscr{W}$, $\mathscr{U}_3$ is the closure of $\mathscr{D}(Y)$ in $\mathscr{U}$, and the distributions of $\mathscr{L}$ and $\mathscr{U}_3$ have their supports in $\bar{Y}$. Moreover, ρ is an isometry from $\mathscr{D}(Y) \subset \mathscr{U}_3$ onto $\mathscr{D}(Y) \subset \mathscr{U}(Y)$: (for $\phi \in \mathscr{D}(Y)$, $\|\phi\|^2_{\mathscr{U}} = \|\phi\|^2_{\mathscr{U}(Y)} = \langle D\phi, \bar{\phi}\rangle$) so, by continuous extension, an isometry from $\mathscr{U}_3$ onto $\mathscr{U}(Y)$.

These developments show that certain classical proofs of the symmetry or positivity of the Green's kernel $G(Y)$ from D into Y are awkward (and in particular involve too many hypotheses): $G(Y)$, the kernel of $\overline{\mathscr{W}(Y)'}$ in $\mathscr{D}'(Y)$, is Hermitian and non-negative.

Notice also that there is no symmetry between the roles of $\mathscr{W}$ and $\mathscr{U}$, nor between those of D and G: D is extendable, G is not. This is what makes $\rho\mathscr{W}$ normal while $\rho\mathscr{U}$ contains $\mathscr{D}(Y)$ but is not normal, so that $\mathscr{W}(Y) = \rho\mathscr{W}$ and $\mathscr{U}(Y) \neq \rho\mathscr{W}$; likewise for $\phi \in \mathscr{D}(Y)$, $\|\phi\|_{\mathscr{U}(Y)} = \|\phi\|_{\mathscr{U}}$, while $\|\phi\|_{\mathscr{W}(Y)} \neq \|\phi\|_{\mathscr{W}}$ (Equation (11.7)) (but $\|D\phi\|_{\mathscr{W}(Y)} = \|D\phi\|_{\mathscr{W}}$, (11.6b)).

Finally, Dirichlet's problem with respect to D and Y is posed as follows. Let us say that a potential $U \in \rho\mathscr{U}$ is "zero on the boundary of Y" if it is an adherent point to $\mathscr{D}(Y)$ in $\rho\mathscr{U}$, i.e. an element of $\mathscr{U}(Y)$. Then, for $T \in \mathscr{W}(Y)$, $U = G(Y)T$ is the unique solution of $DU = T$ in Y, zero on the boundary. If $V \in \rho\mathscr{U}$, the unique solution of $DU = T$, and equal to V on the boundary, is $V + G(Y) \cdot (T - DV)$.

Let us compare these general results to those from Newtonian theory of potentials, corresponding to $D = -\Delta + a$, with Δ Laplacian in $\mathbb{R}^n$ and $a > 0$. Let us take $X = \mathbb{R}^n$, and let us omit it throughout.

We find ourselves exactly in the situation of the Example on page 74; the space $\mathscr{W}$ with kernel $-\Delta + a$ is $\sqrt{a}L^2 + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} L^2 = \mathscr{H}^{-1}$ which is indeed normal, and its anti-dual $\mathscr{U}$ is $\mathscr{H}^1$, with the scalar product

$$(11.8) \qquad (f \mid g)_{\mathscr{H}^1} = \int_{\mathbb{R}^n} \left(af\bar{g} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \overline{\frac{\partial g}{\partial x_i}} \right) dx;$$

$$\|f\|^2_{\mathscr{H}^1} = \int_{\mathbb{R}^n} \left(a|f|^2 + \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right|^2 \right) dx.$$

The kernel of $\mathscr{U}$ in $\mathscr{D}'$ is the Green's operator $G = G_a$ of $-\Delta+a$; $-\Delta+a$ and G_a are the reciprocal canonical isomorphisms between $\mathscr{H}^1$ and $\mathscr{H}^{-1}$, with respect to the norm (11.8) in $\mathscr{H}^1$. The balayage is that defined in the classical theory of potentials. Moreover, in this theory, there exist particular properties: we can define the notion of the capacity of a set, and the notion of properties satisfied almost everywhere in $\mathbb{R}^n$, that is to say, except on a set of zero capacity; a potential $U \in \mathscr{U}$ is a class of functions defined almost everywhere and pairwise almost everywhere equal; the property (Supp) (page 75) is satisfied since $\mathscr{H}^1$ has the approximation property by regularisation. Proposition 32 is extended as follows: T_F^D is *characterised* by the fact that it has support in F, and that its potential is equal to that of T almost everywhere on F; two charges have the same balayage on F if and only if their potentials coincide almost everywhere on F.

Let us now have a look at what happens to the results for open subsets of $\mathbb{R}^n$ in this case. The space $\mathscr{W}(Y)$ relative to Y has the kernel $-\Delta + a$ in $\mathscr{D}'(Y)$, so it is $\mathscr{H}^{-1}(Y)$, which is normal. But here, $\rho\mathscr{U} = \rho(\mathscr{H}^1)$; it is a *subspace* of $\mathscr{H}^1(Y)$, which coincides with $\mathscr{H}^1(Y)$ (as sets but not with the same norm) *only if* Y *is regular enough.* (For example, if the boundary of Y has a compact hypersurface Σ of C^∞-class, and if, at each point of Σ, Y is only on one side of Σ. In the following, we will call this example the regular case). It is the sum $\mathscr{N} + \mathscr{U}(Y)$ of two orthogonal closed subspaces, where $\mathscr{N}$ is the space of a-harmonic functions (satisfying $(-\Delta + a)U = 0$) of $\rho(\mathscr{H})$ and $\mathscr{U}(Y) = (\mathscr{H}^{-1}(Y))' = \mathscr{H}_0^1(Y)$ (the closure of $\mathscr{D}(Y)$ in $\mathscr{H}^1(Y)$) equipped with its usual norm (11.8) (with $\int_Y$ instead of $\int_{\mathbb{R}^n}$). $G(Y)$ is the Green's kernel of $-\Delta + a$ in Y; $-\Delta + a$ and $G(Y)$ are the reciprocal canonical isomorphisms between $\mathscr{H}_0^1(Y)$ and $\mathscr{H}^{-1}(Y)$. $\mathscr{U}_1$ is the subspace of $\mathscr{H}^1(\mathbb{R}^n)$ consisting of functions that vanish on Y; $\mathscr{U}_1 + \mathscr{U}_2$ is the subspace consisting of functions that are a-harmonic in Y; $\mathscr{U}_3 = G\mathscr{L}$ is the subspace of functions that are almost-everywhere zero on $F = Y^c$ [$T \in \mathscr{L}$ if and only if its balayage on F vanishes, i.e. if and only if its potential GT is almost everywhere zero on F]. But the arguments for Y can be applied to $\bar{Y}^c$ in the regular case. Then, just as $\mathscr{U}_3$, the closure of $\mathscr{D}(Y)$ in $\mathscr{U}$, is the space of functions of $\mathscr{H}^1$ almost everywhere zero on $F = Y^c$ or also on $\bar{Y}^c$, $\mathscr{U}_1$, the space of functions of $\mathscr{H}^1$ that vanish on Y, is, in the regular case, the closure of $\mathscr{D}(\bar{Y}^c)$ in $\mathscr{H}^1$; and then, just as $\mathscr{U}_1 + \mathscr{U}_2$, orthogonal to $\mathscr{U}_3$, is the space of functions of $\mathscr{H}^1$ that are a-harmonic in Y, $\mathscr{U}_2 + \mathscr{U}_3$, orthogonal to $\mathscr{U}_1$, is, in the regular case, the space of functions of $\mathscr{H}^1$ that are a-harmonic in $\bar{Y}^c$.

We know that every function in $\mathscr{H}^1$ has a trace on every hypersurface Σ of C^∞-class (it suffices to take its value almost everywhere on Σ): then, in the regular case, $\mathscr{H}_0^1(Y)$ is the subspace of functions of $\mathscr{H}^1(Y)$ with trace almost everywhere zero on Σ, and also of functions which, extended by 0 to Y^c, belong to $\mathscr{H}^1(\mathbb{R}^n)$.

It is also worthwhile to see the relationships between the norms in the two spaces $\mathscr{H}^1(Y)$ and $\rho(\mathscr{H}^1(\mathbb{R}^n))$. Let $f \in \mathscr{H}^1(Y)$. Its norm in $\mathscr{H}^1(Y)$ is given by (11.8) with $\int_Y$ instead of $\int_{\mathbb{R}^n}$. Its norm in $\rho\mathscr{H}^1$ is

$$(11.9) \qquad \left(\int_{\mathbb{R}^n} \left(a \left| \tilde{f} \right|^2 + \sum_{i=1}^n \left| \frac{\partial \tilde{f}}{\partial x_i} \right|^2 \right) dx \right)^{1/2}$$

where $\tilde{f} \in \mathscr{H}^1(\mathbb{R}^n)$ is equal to f in Y, and orthogonal to $\mathscr{U}_1 = \mathscr{H}^1 \cap \rho^{-1}(\{0\})$ in $\mathscr{H}^1$, i.e. it is in $\mathscr{U}_2 + \mathscr{U}_3$, so, in the regular case, a-harmonic in $\bar{Y}^c$. We thus always have

$$\|f\|_{\rho\mathscr{H}^1(\mathbb{R}^n)} = \left\|\tilde{f}\right\|_{\mathscr{H}^1(\mathbb{R}^n)} \geqq \|f\|_{\mathscr{H}^1(Y)}\,; \tag{11.10}$$

moreover, in the regular case, we always have $>$, unless $\tilde{f} = 0$ in $\bar{Y}^c$, i.e. if $f \in \mathscr{H}_0^1(\mathbb{R}^n)$.

Dirichlet's problem indicated on page 78 is the classical Dirichlet's problem for $T \in \mathscr{H}^{-1}(Y)$, and $U = G(Y)T$ is the unique function of $\mathscr{H}_0^1(Y)$ (i.e. in the regular case, the unique function of $\mathscr{H}^1(Y)$ with almost everywhere zero trace on Σ) satisfying $(-\Delta + a)U = T$. The $\tilde{f}$ of (11.9) is, in the regular case, the extension of f in $\mathbb{R}^n$, a-harmonic in $\bar{Y}^c$, and taking the values of f on the boundary Σ of $\bar{Y}^c$ (the exterior Dirichlet problem).

We can study the same problems with $a = 0$. But then $\mathscr{W}$ is only normal in $\mathbb{R}^n$ for $n \geqq 3$ (see Proposition 33 below). Moreover, when it is normal, $\mathscr{W}$ is contained in $\mathscr{H}^{-1}$ but strictly smaller, and $\mathscr{U}$ contains $\mathscr{H}^1$ but is strictly larger. If we only consider $X = \mathbb{R}^n$ for $n \geqq 3$ (or bounded X for any n, because then $\mathscr{W}(X)$ is normal, see Proposition 34), all the results relative to $a > 0$ remain valid for $a = 0$; and, for Y bounded, we still have $\mathscr{W}(Y) = \mathscr{H}^{-1}(Y)$, and $\mathscr{U}(Y) = \mathscr{H}_0^1(Y)$, with the norms (11.8) where $\int_{\mathbb{R}^n}$ is replaced by $\int_Y$, and a by 0 [it suffices indeed to apply the Corollary of Proposition 30, with $u_i = \frac{\partial}{\partial x_i}$ and $\mathscr{H}_i = L^2(Y)$. The classical inequality, for bounded Y:

$$\int_Y |\phi|^2 dx \leqq c(Y) \int_Y \left|\frac{\partial \phi}{\partial x_i}\right|^2 dx \tag{11.11}$$

shows that $(\mathscr{D}(Y))_{\overline{\mathscr{H}(Y)'}}$ (the $\bar{E}_{\bar{H}'}$ of this Corollary) has a continuous inclusion into $\mathscr{D}'(Y)$, and that the norm $\|\cdot\|_{\overline{\mathscr{W}(Y)'}}$ is equivalent to the norm $\|\cdot\|_{\mathscr{H}^1(Y)}$ on $\mathscr{D}(Y)$, hence that the completion $\mathscr{W}(Y)' = \mathscr{U}(Y)$ is $\mathscr{H}_0^1(Y)$].

We said above that, for $D = -\Delta$, $\mathscr{W}$ is only normal in $\mathbb{R}^n$ for $n \geqq 3$. This is a consequence of the following result:

Proposition 33. *Let D be a differential operator with constant coefficients, of type $\geqq 0$. Let $P = \mathscr{F}(D\delta)$ be its associated polynomial, with $P \geqq 0$. For the space $\mathscr{W}$ with kernel D in $\mathscr{D}'(\mathbb{R}^n)$ to be normal, it is necessary and sufficient that $\frac{1}{P}$ is locally integrable and tempered. In this case, if E is the elementary solution of D defined by $\bar{\mathscr{F}}\frac{1}{P}$, the kernel of the space $\mathscr{U}$ of potentials is $G = E*$, i.e. $\phi \mapsto E * \phi$.*

Proof. The operator D, as it has constant coefficients, is invariant under translations of $\mathbb{R}^n$, operating on $\mathscr{D}'(\mathbb{R}^n)$ and $\mathscr{D}(\mathbb{R}^n)$; so the space $\mathscr{W}$, which is defined by D in $\mathscr{D}'(\mathbb{R}^n)$, is also invariant under translations of $\mathbb{R}^n$ (invariance under an automorphism, page 46).

1°) Let us then suppose that $\mathscr{W}$ is normal. Its anti-dual $\bar{\mathscr{W}}' \subset \mathscr{D}'$ is also invariant under translations of $\mathbb{R}^n$ (we use here the fact that these translations, operating on $\mathscr{D}$ and $\mathscr{D}'$, leave the inclusion I of $\mathscr{D}$ in $\mathscr{D}'$ invariant); so its kernel G is a continuous linear map from $\mathscr{D}$ into $\mathscr{D}'$, commuting with the translations of $\mathbb{R}^n$, and so is a convolution $\phi \mapsto E * \phi$, with $E \in \mathscr{D}'(\mathbb{R}^n)$.

By Proposition 28b, we necessarily have $DE = \delta$; moreover, E is of positive type in the sense of Bochner, as a result of G being a non-negative kernel: for every $\phi \in \mathscr{D}$, we have

$$\langle E * \phi, \bar{\phi} \rangle = \langle G\phi, \bar{\phi} \rangle \geqq 0. \tag{11.12}$$

The Fourier image of $D\delta$ is a polynomial P, and the positivity of D immediately implies that $P \geqq 0$; $\mathscr{F}E$ is a measure $\mu \geqq 0$, and $DE = \delta$ implies $P\mu = 1$, i.e. $P\mu = d\xi$, the Lebesgue measure; so $\mu = \frac{1}{P} d\xi$ in the complement of the manifold of the zeros of P; this proves that $\frac{1}{P}$ is locally integrable on $\mathbb{R}^n$. And then, we necessarily have $\mu = \frac{1}{P} d\xi + v$ with $v \geqq 0$, carried over by the manifolds of the zeros of P: μ must be tempered, and so $\frac{1}{P}$ and v must also be tempered.

Let us remark that conversely, for every measure μ of this form, we have $P\mu = 1$; so $E = \bar{\mathscr{F}}\mu$ is a distribution of type $\geqq 0$ such that $DE = \delta$, and $G : \phi \mapsto E * \phi$ is a non-negative kernel that is an inverse of D, so we necessarily have $v = 0$, and $E = \bar{\mathscr{F}} \frac{1}{P}$, $G = E*$.

2°) Conversely, if $\frac{1}{P}$ is locally integrable and tempered, and if $E = \bar{\mathscr{F}} \frac{1}{P}$, $G = E*$ is a non-negative kernel that is an inverse of D, then by Proposition 31, $\mathscr{W}$ is normal.

□

Example. $D = -\Delta$, and $P(\xi) = 4\pi^2 |\xi|^2$ (where $|\xi| = \left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2}$). Then $\frac{1}{4\pi^2 |\xi|^2}$ is locally integrable if and only if $n \geqq 3$, and then it converges to 0 at infinity, so is tempered: $\mathscr{W}$ is normal for $n \geqq 3$. In this case, $G = E*$ with

$$E = \bar{\mathscr{F}} \frac{1}{4\pi^2 |\xi|^2} = \frac{1}{(n-2) S_n} \frac{1}{|\xi|^{n-2}}, \tag{11.13}$$

where S_n is the area of the unit sphere in $\mathbb{R}^n$.

It is classical to use this kernel G for the theory of Newtonian potentials with $-\Delta$. There are infinitely many non-negative inverses of $-\Delta$, of the form $G + N$, with $N \geqq 0$ and $\Delta \circ N = 0$; Proposition 31 tells us why we have to take G itself, the smallest, since we want $\mathscr{U} = \bar{\mathscr{W}}'$, where $\mathscr{W}$ is normal. Let us now take the operator $-\Delta$ in $\mathbb{R}^2$, for example. In the classical potential theory, there is always some confusion as to how to define $\mathscr{W}$; this is because we try to define $\mathscr{W}$ from G, the kernel of $\bar{\mathscr{W}}'$ (Proposition 29), which is an annoying bidualisation when $\bar{\mathscr{W}}'$ does not exist as a subspace of $\mathscr{D}'$! In practice, $\mathscr{W}$ does still exist, as a space associated to $-\Delta$, *but it is no longer normal*, $\mathscr{D} \not\subset \mathscr{W}$. Let us see what is going on. $\mathscr{F}\mathscr{W}$ has multiplication by $P = 4\pi^2 |\xi|^2$ as its kernel. So (Proposition 22) $\mathscr{F}\mathscr{W} = 2\pi |\xi| L^2$. Let $\phi \in \mathscr{D}(\mathbb{R}^n)$ and $\hat{\phi} = \mathscr{F}\phi$. Then $\hat{\phi} \in 2\pi |\xi| L^2$ equals $\frac{\hat{\phi}(\xi)}{|\xi|} \in L^2$, where $\hat{\phi}(0) = 0$ (taking into account that $\hat{\phi}$ is analytic and is in L^2), or $\int_{\mathbb{R}^n} \phi(x) dx = 0$: $\mathscr{W} \cap \mathscr{D}$ is the subspace of $\mathscr{D}$ consisting of functions that integrate to 0, which is a hyperplane in $\mathscr{D}$.

Proposition 34. *Let D be a differential operator with constant coefficients of positive type on $\mathbb{R}^n$. For every non-empty bounded open subset X of $\mathbb{R}^n$,* *the Hilbert subspace $\mathscr{W}(X)$, with kernel D relative to $\mathscr{D}'(X)$, is normal; moreover, $\mathscr{U}(X) \subset L^2 \subset \mathscr{W}(X)$. Let D' be a differential operator with constant coefficients; let $P = \mathscr{F}(D\delta)$ and $P' = \mathscr{F}(D'\delta)$. For $D'\mathscr{U}(X) \subset \mathscr{W}(X)$, it is necessary and sufficient that P' is weaker than P in the sense of Hörmander*(57).

We will prove this Proposition at the same time as the others in a later study on Hilbert spaces associated to differential operators of type $\geqq 0$.

But this Proposition suffices to show that there exist theories of potentials and Dirichlet's problems for bounded open subsets of $\mathbb{R}^n$, for differential operators very far from being elliptic: thus $\Box^2$, where $\Box$ is the wave operator $\frac{\partial^2}{\partial x_n^2} - \frac{\partial^2}{\partial x_1^2} \cdots - \frac{\partial^2}{\partial x_{n-1}^2}$, is hyperbolic, and its Cauchy problem is well-posed (and has classical solution) with respect to the time variable $t = x_n$; however, it is of type $\geqq 0$ (like every operator D^*D), so it has a Dirichlet problem for every bounded open set. The key is to know how to interpret it well; the spaces $\mathscr{W}$ and $\mathscr{U}$ give the correct solution of the problem.

§12. Hermitian subspaces and associated Hermitian kernels

Let us return to the results at the end of §8. The cone $\mathrm{Hilb}(E)$ associated to a locally convex, quasi-complete Hausdorff topological vector space E is regular, so generates a vector space over $\mathbb{R}$, which we will denote by $\mathrm{Hilb}(E)\otimes\mathbb{R}$ (strictly speaking, this is not a tensor product!). An element of $\mathrm{Hilb}(E) \otimes \mathbb{R}$ can be written, in infinitely different ways, as a formal difference $\mathscr{H}_1 - \mathscr{H}_2$ of two Hilbert subspaces of E, with the equivalence relation $\mathscr{H}_1 - \mathscr{H}_2 \simeq \mathscr{H}_3 - \mathscr{H}_4$ if $\mathscr{H}_1 + \mathscr{H}_4 = \mathscr{H}_2 + \mathscr{H}_3$. Likewise, $\mathscr{L}^+(E)$ generates a vector space over $\mathbb{R}$, which we denote by $\mathscr{L}^+(E) \otimes \mathbb{R}$, and which is the space of Hermitian kernels relative to E, which can be expressed as a difference of two non-negative elements. The isomorphism $\mathrm{Hilb}(E) \to \mathscr{L}^+(E)$ extends in a unique manner to an isomorphism $\mathrm{Hilb}(E)\otimes\mathbb{R} \to \mathscr{L}^+(E)\otimes\mathbb{R}$, which is again functorial. Our goal, in this section, is to express the elements of $\mathrm{Hilb}(E)\otimes\mathbb{R}$ as vector subspaces or classes of vector subspaces of E.

A pre-Hermitian space $\mathscr{H}$ is a (non-topological) vector space over $\mathbb{C}$ equipped with a Hermitian form denoted by $(h, k) \mapsto (h \mid k)_{\mathscr{H}}$. If this form is non-negative, it is a pre-Hilbert space; it can then be equipped with a norm and hence a topology, for which the Hermitian form is continuous; no such analogues exist here. The form defines a linear map γ from $\mathscr{H}$ into its algebraic anti-dual $\bar{\mathscr{H}}^*$, given by

$$(12.1) \qquad (h \mid k)_{\mathscr{H}} = (h \mid \gamma k)_{\mathscr{H}, \bar{\mathscr{H}}^*}.$$

We say that h and k are orthogonal if $(h \mid k) = 0$, which is equivalent to $(k \mid h)_{\mathscr{H}} = 0$ because the form is Hermitian. The form is said to be non-degenerate if the orthogonal complement of $\mathscr{H}$ is $\{0\}$, in other words, if γ is injective.

Let, moreover, $\mathscr{H}$ be a vector subspace of E. which we recall is a locally convex, quasi-complete Hausdorff topological vector space. The inclusion $j : \mathscr{H} \to E$ has an adjoint $j^* : E^* \to \bar{\mathscr{H}}^*$, so a fortiori $\bar{E}' \to \bar{\mathscr{H}}^*$. An *admissible pre-Hermitian subspace of* E is a vector subspace $\mathscr{H}$ of E with a *non-degenerate*

(57) Hörmander [1], 2nd Part, Chapter III, 3.3, page 71.

Hermitian form on $\mathscr{H}$, denoted $(\cdot \mid \cdot)_{\mathscr{H}}$, such that we have

$$j^*(\bar{E}') \subset \gamma\mathscr{H}. \tag{12.2}$$

For example, let $\mathscr{K}$ is a Hilbert subspace of E, with associated kernel K. Let $\mathscr{H}$ be a dense pre-Hilbert subspace of $\mathscr{K}$, with the induced structure. Let i and k be the inclusions $\mathscr{H} \xrightarrow{i} \mathscr{K}$ and $\mathscr{K} \xrightarrow{k} E$, such that the inclusion $j : \mathscr{H} \to E$ is ki. The map $\gamma_{\mathscr{K}}$ relative to $\mathscr{K}$ is an isomorphism from $\mathscr{K}$ onto $\bar{\mathscr{K}}'$; the map $\gamma_{\mathscr{H}}$ relative to $\mathscr{H}$ is $i^*\gamma_{\mathscr{K}}i$; but, as $\mathscr{H}$ is dense in $\mathscr{H}$, $\mathscr{H}$ and $\mathscr{K}$ have the same dual, so that $i^* : \mathscr{K}^* \to \bar{\mathscr{H}}^*$ is the identity $\bar{\mathscr{K}}' \to \bar{\mathscr{H}}'$; if we identify $\bar{\mathscr{K}}'$ and $\bar{\mathscr{H}}'$, $i^*\gamma_{\mathscr{K}}i$ is simply the restriction of $\gamma_{\mathscr{K}}$ to $\mathscr{H}$. The kernel K associated to $\mathscr{K}$ in E is θk^* where θ is $\gamma_{\mathscr{K}}^{-1} : \bar{\mathscr{K}}' \to \mathscr{K}$; the condition (12.2) relative to $\mathscr{H}$ here translates to $k^*(\bar{E}') \subset \gamma_{\mathscr{K}}(\mathscr{H})$, where $K(\bar{E}') \subset \mathscr{H}$: a pre-Hilbert subspace $\mathscr{H}$ of $\mathscr{K}$, with the induced structure, is admissible if and only if it contains $K(\bar{E}')$. We will later define a kernel associated to each admissible pre-Hermitian subspace: all the $\mathscr{H}$ such that $K(\bar{E}') \subset \mathscr{H} \subset \mathscr{K}$ will have the same kernel K.

A priori, it may appear annoying to introduce such a large family of subspaces, thus taking away any hope of having a *bijective* correspondence between subspaces and kernels; but we will see later that, even with the most restrictive sense which we can give to the Hermitian subspaces, such a bijective correspondence does not exist; and the category defined here will turn out to be very practical for the statements and proofs.

If, then, $\mathscr{H}$ is an admissible pre-Hermitian subspace of E, and if θ is the map $\gamma(\mathscr{H}) \to \mathscr{H}$ which is inverse of the bijection $\gamma : \mathscr{H} \to \gamma(\mathscr{H})$, we can define $H = j\theta j^*$ by the hypothesis $j^*(\bar{E}') \subset \gamma(\mathscr{H})$. H is a linear map from $\bar{E}'$ into E, and even from $\bar{E}'$ into $\mathscr{H}$, if, as already done on page 19, we adopt the convention of identifying $j\theta j^*$ and θj^*.

Proposition 35. *Let $\mathscr{H}$ be an admissible pre-Hermitian subspace of E. The map $H : \bar{E}' \to E$ defined above by $j\theta j^*$ is a Hermitian kernel. It satisfies (4.2) and (4.3). Conversely, if $\mathscr{H}$ is a vector subspace of E, if it is equipped with a non-degenerate Hermitian form, and if there exists a linear map $H : \bar{E}' \to \mathscr{H}$ satisfying (4.2), $\mathscr{H}$ is an admissible pre-Hermitian subspace of E, and H is its kernel. Finally, every Hermitian kernel $H : \bar{E}' \to E$ is associated to at least one admissible pre-Hermitian subspace.*

Proof. (4.2) can be shown as in Proposition 6. We deduce (4.3) from it, so H is Hermitian, and thus is a Hermitian kernel by Proposition 4.

Conversely, let $\mathscr{H} \subset E$, equipped with a non-degenerate Hermitian form and let us suppose that there exists a map $H : \bar{E}' \to E$ satisfying (4.2).

The relationship $(h \mid H\bar{e}')_{\mathscr{H}} = (h \mid \gamma H\bar{e}')_{\mathscr{H},\bar{\mathscr{H}}'} = (jh \mid \bar{e}')_{E,\bar{E}'} = (h \mid j^*\bar{e}')_{E,\bar{E}}$ shows that $\gamma H = j^*$, so $j^*(\bar{E}') \subset \gamma(\mathscr{H})$, so $\mathscr{H}$ is an admissible pre-Hermitian subspace of E. Then, as γ is injective, $\gamma H = j^*$ gives $H = \theta j^*$, and H is the kernel of $\mathscr{H}$ in E.

Finally, let H be a Hermitian kernel: $\bar{E}' \to E$. Then $\mathscr{H} = H(\bar{E}')$ is a vector subspace of E, equipped with the Hermitian form (4.3) (we saw in the proof of Proposition 16 that this form is well-defined). This form is non-degenerate: if $H\bar{e}'$ is such that, for every $f' \in E'$, $(H\bar{e}' \mid H\bar{f}')_{\mathscr{H}} = 0$, we also have $\langle He', f' \rangle = 0$ so $H\bar{e}' = 0$. For $e' \in E'$, $j^*\bar{e}'$ is the antilinear form on $\mathscr{H}$ defined by the restriction to $\mathscr{H}$ of the antilinear form defined by $\bar{e}'$ on E; the equality (4.2) then shows that $\gamma H\bar{e}' = j^* e'$, so we have here exactly $j^*(\bar{E}') = \gamma(\mathscr{H})$; $H(\bar{E}')$ is an admissible pre-Hermitian subspace and then (4.2) precisely shows that its associated kernel is H. $(H\bar{E}')$ is the smallest (in the set-theoretic sense) admissible pre-Hermitian subspace of E with the given Hermitian kernel H. □

Now let $\mathscr{H}$ be an admissible pre-Hermitian subspace of E, and u a weakly continuous linear map from E into F; let us try to define an admissible pre-Hermitian image space $u(\mathscr{H})$. One could be tempted to say that, as a set, it is the image of $\mathscr{H}$ under u; and then define a pre-Hermitian structure on this image; but we will not be able to do this. On the contrary, let us go back to the proof of Proposition 21. Let $\mathscr{N} = \mathscr{H} \cap u^{-1}(\{0\})$; it is a vector subspace of $\mathscr{H}$. Let $\mathscr{K}$ be the orthogonal complement of $\mathscr{N}$ in $\mathscr{H}$, with respect to the Hermitian form of $\mathscr{H}$. *We do not necessarily have* $\mathscr{N} + \mathscr{K} = \mathscr{H}$. Let us then consider the space $u(\mathscr{K}) \subset F$. The map $u : \mathscr{K} \to F$ factorises as

$$\mathscr{K} \to \mathscr{K}/(\mathscr{K} \cap \mathscr{N}) \xrightarrow{\dot{u}} F, \qquad \dot{u} \text{ injective.}$$

If we equip $\mathscr{K}$ with the restriction of the Hermitian form to $\mathscr{H}$, we obtain a potentially degenerate Hermitian form, whose kernel (the subspace orthogonal to the whole space) is $\mathscr{K} \cap \mathscr{N}$. Then, on the quotient $\mathscr{K}/(\mathscr{K} \cap \mathscr{N})$, we can define a non-degenerate Hermitian form (it is something we could not have done if we had taken $\mathscr{H}$ instead of $\mathscr{K}$). We can transport it onto $u(\mathscr{K})$ under $\dot{u}$, which is thus equipped with a non-degenerate Hermitian form; for $x, y \in \mathscr{K}$, we have

$$(12.3) \qquad (x \mid y)_{\mathscr{H}} = (x \mid y)_{\mathscr{K}} = (u(x) \mid u(y))_{u(\mathscr{K})}.$$

Then let $h = u(k) \in u(\mathscr{K})$ and $f' \in F'$;

$$(12.4) \qquad \begin{aligned} \langle h, f' \rangle_{F,F'} &= (h \mid \bar{f}')_{F,\bar{F}'} = (u(k) \mid \bar{f}')_{F,\bar{F}'} \\ &= (k \mid u^* \bar{f}')_{E,\bar{E}'} = (k \mid Hu^* \bar{f}')_{\mathscr{H}}. \end{aligned}$$

If $k \in \mathscr{N}$, $h = u(k) = 0$, so all these quantities vanish; so $Hu^* f'$ is orthogonal to $\mathscr{N}$ in $\mathscr{H}$, i.e. it is always an element of $\mathscr{K}$, whatever f' is. The last scalar product in (12.4) is thus a scalar product in $\mathscr{K}$, and then (12.3) shows that it is equal to

$$(12.5) \qquad \langle h, f' \rangle_{F,F'} = (u(k) \mid uHu^* \bar{f}')_{u(\mathscr{K})} = (h \mid uHu^* \bar{f}')_{u(\mathscr{K})}.$$

Then Proposition 35 shows that $u(\mathscr{K})$ is an admissible pre-Hermitian subspace of F, with associated kernel uHu^*. It is $u(\mathscr{K})$ that we will call *the image of* $\mathscr{H}$ *under* u, and we will denote it by $\ddot{u}(\mathscr{H})$. It is contained in the set-theoretic image $u(\mathscr{H})$, but is in general smaller, which is why we use a different notation $\ddot{u}$ instead of u (whereas we could use u for Hilbert subspaces).

For example, it could happen that $\mathscr{K} = \mathscr{N}$ (we will see this below on page 85), so the set $u(\mathscr{H})$ could be anything but the pre-Hermitian subspace $\ddot{u}(\mathscr{H})$ of F would be $\{0\}$. If $\mathscr{H}$ is a Hilbert subspace of E, the $\ddot{u}(\mathscr{H})$ that we just defined coincides with $u(\mathscr{H})$ of §8. If, in particular, E is a vector subspace of F, equipped with a finer topology than the induced topology, every admissible pre-Hermitian subspace of E is a fortiori an admissible pre-Hermitian subspace of F.

For an admissible pre-Hermitian subspace $\mathscr{H}$ of E, and a *real* scalar λ, we will define $\lambda\mathscr{H}$ as $\{0\}$ if $\lambda = 0$, and if $\lambda \neq 0$, as the same space $\mathscr{H}$ equipped with a new Hermitian form, obtained by multiplying the old one by $\frac{1}{\lambda}$. If H is the kernel of $\mathscr{H}$, that of $\lambda\mathscr{H}$ is λH.

Finally, let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two admissible pre-Hermitian subspaces of E, with associated kernels H_1 and H_2. Then, $\mathscr{H}_1 \times \mathscr{H}_2 = \mathscr{H}_1 \oplus \mathscr{H}_2$, equipped with the Hermitian form $((x_1, x_2) \mid (y_1, y_2))_{\mathscr{H}_1 \oplus \mathscr{H}_2} = (x_1 \mid y_1)_{\mathscr{H}_1} + (x_2 \mid y_2)_{\mathscr{H}_2}$, is trivially an admissible pre-Hermitian subspace of $E \times E$. But $(x, y) \to x + y$ is a continuous linear map Φ from $E \times E$ into E; the image $\ddot{\Phi}(\mathscr{H}_1 \oplus \mathscr{H}_2)$, in the sense of the image $\ddot{u}(\mathscr{H})$ in the preceding pages, will be called the admissible pre-Hermitian subspace $\mathscr{H}_1 \ddot{+} \mathscr{H}_2$, the sum of $\mathscr{H}_1$ and $\mathscr{H}_2$. We obtain it as follows. The kernel $\mathscr{N}$ of Φ in $\mathscr{H}$ is the set of $(k_1, k_2) \in \mathscr{H}_1 \oplus \mathscr{H}_2$ such that $k_1 + k_2 = 0$. Let $\mathscr{K}$ be the orthogonal complement of $\mathscr{N}$; the points $(H_1\bar{e}', H_2\bar{e}')$ with $e' \in E'$ are in $\mathscr{K}$ (see the proof of Proposition 12). Then the restriction to $\mathscr{K}$ of the Hermitian form of $\mathscr{H}_1 \oplus \mathscr{H}_2$ has kernel $\mathscr{K} \cap \mathscr{N}$, and so defines a non-degenerate Hermitian form on $\mathscr{K}/(\mathscr{K} \cap \mathscr{N})$; the map Φ passes to the quotient: $\mathscr{K}/(\mathscr{K} \cap \mathscr{N}) \xrightarrow{\Phi^\bullet} E$, and $\mathscr{H}_1 \ddot{+} \mathscr{H}_2$ is $\Phi^\bullet(\mathscr{K}/(\mathscr{K} \cap \mathscr{N})) = \Phi(\mathscr{K})$, with the transported Hermitian form. The proof on page 13 shows that, if $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert subspaces of E, the definition given here for $\mathscr{H}_1 \ddot{+} \mathscr{H}_2$ gives again the old definition of $\mathscr{H}_1 + \mathscr{H}_2$. The adjoint of $\Phi : (e_1, e_2) \mapsto (e_1 + e_2)$ from $E \times E$ into E is $\Phi^* : \bar{e}' \mapsto (\bar{e}', \bar{e}')$ from $\bar{E}'$ into $\bar{E}' \times \bar{E}'$ (diagonal map); the kernel associated to $\mathscr{H}_1 \oplus \mathscr{H}_2$ in $E \times E$ is $(H_1, H_2) : (\bar{e}'_1, \bar{e}'_2) \mapsto (H_1\bar{e}'_1, H_2\bar{e}'_2)$; so the kernel associated to $\mathscr{H}_1 + \mathscr{H}_2$ in E is $\Phi \circ (H_1, H_2) \circ \Phi^* = H_1 + H_2 : \bar{e}' \mapsto H_1\bar{e}' + H_2\bar{e}'$. We could also have directly repeated the proof of Proposition 12.

Just as $\ddot{u}(\mathscr{H})$ can, as a set, be smaller than $u(\mathscr{H})$, $\mathscr{H}_1 \ddot{+} \mathscr{H}_2$ can be smaller than $\mathscr{H}_1 + \mathscr{H}_2$. Let us take, for example, $\mathscr{H}_1 = \mathscr{H}$ and $\mathscr{H}_2 = -\mathscr{H}$. Here, the kernel $\mathscr{N}$ of the map Φ is the set of $(h, -h) \in \mathscr{H} \oplus (-\mathscr{H})$. Its orthogonal complement $\mathscr{K}$ is $\mathscr{N}$ itself, because $((k_1, k_2) \mid (h, -h))_{\mathscr{H} \oplus (-\mathscr{H})} = ((k_1 + k_2) \mid h)_{\mathscr{H}}$ vanishes if and only if $k_2 = -k_1$. Then, the set $\mathscr{H} + (-\mathscr{H})$ is $\mathscr{H}$, while $\mathscr{H} \ddot{+} (-\mathscr{H}) = \{0\}$.

The sum $\mathscr{H}_1 \ddot{+} (-\mathscr{H}_2)$ will also be denoted $\mathscr{H}_1 \ddot{-} \mathscr{H}_2$, or even $\mathscr{H}_1 - \mathscr{H}_2$, as the ambiguity with the set-theoretic difference is hardly problematic. If $\mathscr{H}_1$ and $\mathscr{H}_2$ have intersection $\{0\}$, then $\mathscr{N} = \{0\}$, $\mathscr{K} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and $\mathscr{H} \ddot{+} \mathscr{H}_2$ is nothing but the vector subspace $\mathscr{H}_1 + \mathscr{H}_2$ of E, with the Hermitian form given by the direct sum of the forms given on $\mathscr{H}_1$ and $\mathscr{H}_2$; $\mathscr{H}_1$ and $\mathscr{H}_2$ are orthogonal in $\mathscr{H}_1 \ddot{+} \mathscr{H}_2$. In this case, we will also write $\mathscr{H}_1 + \mathscr{H}_2$ instead of $\mathscr{H}_1 \ddot{+} \mathscr{H}_2$.

We should beware of thinking that multiplication by scalars and image under a map possess good properties that we expect of them. For example, addition, which is commutative, *is not associative*: $(\mathscr{H}_1 \ddot{+} \mathscr{H}) \ddot{+} \mathscr{H}_3 \neq \mathscr{H}_1 \ddot{+} (\mathscr{H}_2 \ddot{+} \mathscr{H}_3)$. This is why it is necessary now to establish an equivalence relation between admissible pre-Hermitian subspaces of E. We will write $\mathscr{H}_1 \simeq \mathscr{H}_2$ if $\mathscr{H}_1 - \mathscr{H}_2 = \{0\}$ (we then have $\mathscr{H} \simeq \mathscr{H}$, as we just saw above).

Proposition 36. *The three operations* $(\lambda, \mathscr{H}) \mapsto \lambda\mathscr{H}$, $(\mathscr{H}_1, \mathscr{H}_2) \mapsto \mathscr{H}_1$ *and* $\mathscr{H} \mapsto \ddot{u}(\mathscr{H})$ *are assumed to be defined as above. Then* $\mathscr{H}_1 \simeq \mathscr{H}_2$ *if* $\mathscr{H}_1 - \mathscr{H}_2 = \{0\}$ *is an equivalence relation; if* H_1 *and* H_2 *are the kernels associated to* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *relative to* E, *we have* $\mathscr{H}_1 \simeq \mathscr{H}_2$ *if and only if* $H_1 = H_2$. *The three operations defined above pass to the quotient by the equivalence relation: the set* $\mathrm{Herm}(E)$ *of equivalence classes of admissible pre-Hermitian subspaces of* E, *equipped with the laws of multiplication by real scalars and addition, is a vector* *space over* $\mathbb{R}$*. The map* $\mathscr{H}^\bullet \mapsto H$ *(where* $\mathscr{H}^\bullet$ *is the class to which* $\mathscr{H}$ *belongs, and* H *is the kernel associated to* $\mathscr{H}$*) is a linear bijection from* $\mathrm{Herm}(E)$ *onto the vector space (over* $\mathbb{R}$*)* $\mathscr{L}^h(E)$ *of Hermitian kernels* $\bar{E}' \to E$*. Finally, if* u *is a weakly continuous linear map from* E *into* F*,* $\mathscr{H}^\bullet \mapsto (\ddot{u}(\mathscr{H}))^\bullet$ *is a linear map from* $\mathrm{Herm}(E)$ *into* $\mathrm{Herm}(F)$*, associated by this bijection to the linear map* $H \mapsto uHu^*$ *from* $\mathscr{L}^h(E)$ *into* $\mathscr{L}^h(F)$*. The functors* Herm *and* $\mathscr{L}^h$ *from the category of locally convex, quasi-complete Hausdorff topological vector spaces (over* $\mathbb{C}$*) into the category of vector spaces over* $\mathbb{R}$ *are isomorphic.*

Proof. The kernel associated with $\mathscr{H}_1 - \mathscr{H}_2$ is $H_1 - H_2$. This admissible pre-Hermitian subspace of E is $\{0\}$ if and only if its associated kernel is zero. [Obviously, the kernel of $\{0\}$ is zero. Conversely, if an admissible pre-Hermitian subspace $\mathscr{H}$ has 0 as its kernel, we have $j\theta j^* = 0$; but j and θ are injective, so $j^* = 0$, so $j = 0$ and $\mathscr{H} = \{0\}$.] So we indeed have $\mathscr{H}_1 \simeq \mathscr{H}_2$ if and only if $H_1 = H_2$. This immediately shows that it is an equivalence relation; moreover, we see that this equivalence relation is compatible with the operations under consideration, which thus pass to the quotient. Furthermore, the map $\mathscr{H}^\bullet \mapsto H$ is a bijection from the quotient $\mathrm{Herm}(E)$ onto the vector space $\mathscr{L}^h(E)$ (Proposition 35); this bijection respects addition and multiplication by real scalars, so $\mathrm{Herm}(E)$ is also a vector space over $\mathbb{R}$ and its bijection onto $\mathscr{L}^h(E)$ is linear. The end of the Proposition is obvious (see the end of §8). □

Proposition 37. *Let* $\mathscr{H}$ *be an admissible pre-Hermitian subspace of* E*. Let us assume that there exists a Hilbert structure on* $\mathscr{H}$*, with respect to which the given Hermitian form is continuous. Then* $\mathscr{H}$ *admits a decomposition* $\mathscr{H} = \mathscr{H}_1 - \mathscr{H}_2$*, where* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *are admissible pre-Hilbert spaces, with trivial intersection (so orthogonal in* $\mathscr{H}$*). We thus also have a decomposition of its kernel,* $H = H_1 - H_2$*, where* H_1 *and* H_2 *are positive kernels.*

Proof. Let us denote by $((\cdot \mid \cdot))$ the Hilbert scalar product, assumed to exist on $\mathscr{H}$, and $(\cdot \mid \cdot)_{\mathscr{H}}$ the given Hermitian form, assumed to be continuous. We then know that there exists a continuous Hermitian operator A on $\mathscr{H}$, such that

$$(h \mid k)_{\mathscr{H}} = ((h \mid Ak)). \tag{12.6}$$

By the spectral decomposition $A = A^+ - A^-$, we can find two closed vector subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ of $\mathscr{H}$, with $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$ and $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$, that are orthogonal with respect to $((\cdot \mid \cdot))$, and stable under A and so orthogonal with respect to $(\cdot \mid \cdot)_{\mathscr{H}}$. Moreover, we have $A \geqq 0$ on $\mathscr{H}_1$, and hence $(\cdot \mid \cdot)_{\mathscr{H}} \geqq 0$ on $\mathscr{H}_1$, hence A is positive definite, because it is non-degenerate on $\mathscr{H}$ and hence on $\mathscr{H}_1$ and $\mathscr{H}_2$; and A is negative definite on $\mathscr{H}_2$. Let us show that $\mathscr{H}_1$ and $-\mathscr{H}_2$ are admissible pre-Hilbert subspaces. The decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ gives the dual decomposition $\bar{\mathscr{H}}^* = \bar{\mathscr{H}}_1^* \oplus \bar{\mathscr{H}}_2^*$, and, since they are orthogonal with respect to $(\cdot \mid \cdot)_{\mathscr{H}}$, $\gamma = \gamma_1 + \gamma_2$, where γ_1 and γ_2 are associated with the restrictions of the Hermitian form to $\mathscr{H}_1$ and $\mathscr{H}_2$. Then $\gamma(\mathscr{H}) = \gamma_1(\mathscr{H}_1) \oplus \gamma_2(\mathscr{H}_2)$. If i_1 is the inclusion $\mathscr{H}_1 \to \mathscr{H}$, i_1^* is the projection $pr_1 : \bar{\mathscr{H}}^* \to \bar{\mathscr{H}}_1^*$; the inclusion $J_1 : \mathscr{H}_1 \to E$ is ji, so J_1^* is $i_1^* j^* = pr_1 \circ j^*$. Then, from $j^*(\bar{E}') \subset \gamma(\mathscr{H}) = \gamma_1(\mathscr{H}_1) \oplus \gamma_2(\mathscr{H}_2)$, we immediately deduce $J_1^*(\bar{E}') = pr_1 j^*(\bar{E}') \subset \gamma_1(\mathscr{H}_1)$: $\mathscr{H}_1$ is admissible, and likewise for $-\mathscr{H}_2$; and we indeed have $\mathscr{H} = \mathscr{H}_1 - \mathscr{H}_2$. We deduce from this that, for the kernels, $H = H_1 - H_2$. □

Remark 1. The conditions given in the statement are therefore sufficient for $\mathscr{H}$ to admit a decomposition into $\mathscr{H}_1 - \mathscr{H}_2$; they cannot be necessary, because of the too large generality of admissible subspaces. Nevertheless, we will remark that the category of subspaces satisfying these conditions are stable with respect to the three operations; it is obvious for multiplication by reals, and, if we show it for the image $\ddot{u}$ under $u : E \to F$, it will also be true for the addition $\ddot{+}$. Yet, if we return to the developments on pages 84, we see that, with respect to the Hilbert structure $((\cdot \mid \cdot))$ of $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{N}$ are closed since $(\cdot \mid \cdot)_{\mathscr{H}}$ is continuous; so the Hermitian form $(\cdot \mid \cdot)_{\mathscr{H}}$ on $\mathscr{K}$ is continuous, with kernel $\mathscr{K} \cap \mathscr{N}$, and as a consequence, the non-degenerate Hermitian form on $\mathscr{K}/(\mathscr{K} \cap \mathscr{N})$ is continuous with respect to the quotient Hilbert structure. We could therefore have taken these properties stated in Proposition 37 as the definition of an admissible subspace, then pass to the quotient by the same equivalence relations; the quotient obtained would have been the vector subspace (over $\mathbb{R}$) of $\mathrm{Herm}(E)$ produced by the cone $\mathrm{Hilb}(E)$ (we can identify $\mathrm{Hilb}(E)$ with its image in $\mathrm{Herm}(E)$; indeed, passing to the quotient $\mathscr{H} \mapsto \mathscr{H}^\bullet$ is injective for the set of Hilbert subspaces, since $\mathscr{H} \to H$ is injective), since its image in $\mathscr{L}^h(E)$ under the canonical bijection $\mathscr{H}^\bullet \mapsto H$ is the vector subspace of $\mathscr{L}^h(E)$ generated by $\mathscr{L}^+(E)$.

Remark 2. Let $\mathscr{H}$ have the properties of the statement of Proposition 37; since $H = H_1 - H_2$, we also have a decomposition $\mathscr{H} \simeq \hat{\mathscr{H}_1} - \hat{\mathscr{H}_2}$, where $\hat{\mathscr{H}_1}$ and $\hat{\mathscr{H}_2}$ are two *Hilbert* subspaces with kernels H_1 and H_2; these spaces are not necessarily $\mathscr{H}_1$ and $\mathscr{H}_2$ themselves, which are only pre-Hilbert. We therefore do not have $\mathscr{H} = \hat{\mathscr{H}_1} - \hat{\mathscr{H}_2}$. In certain particular spaces, $\mathrm{Herm}(E)$ and $\mathscr{L}^h(E)$ are generated by $\mathrm{Hilb}(E)$ and $\mathscr{L}^+(E)$. This is the case, for example, if E admits a Hilbert structure. Indeed, in this case, we can identify $\bar{E}'$ with E; a Hermitian kernel H is then a continuous Hermitian operator from E into E, and it is always the difference of two non-negative kernels. But this represents, without doubt, a very exceptional case; in any case, we will later (page 88) give examples showing that this is not always the case.

Proposition 38. *Let H be a Hermitian kernel: $\bar{E}' \to E$. For it to be the difference of two non-negative kernels, it is necessary and sufficient that it is upper-bounded by a non-negative kernel, in other words, that there exists a kernel $L \geqq 0$ such that*

$$\big|\langle H\bar{e}', f'\rangle\big| \leqq \langle L\bar{e}', e'\rangle^{1/2} \langle L\bar{f}', f'\rangle^{1/2} \tag{12.7}$$

or

$$\big|\langle H\bar{e}', e'\rangle\big| \leqq \langle L\bar{e}', e'\rangle. \tag{12.8}$$

If this is the case, H is the difference of two non-negative alien kernels.

Proof. (12.7) and (12.8) are equivalent, by the Lemma on page 30. If $H = H_1 - H_2$, we do have these inequalities with $L = H_1 + H_2$; conversely, (12.8) implies that the kernel $L - H$ is non-negative, and then $H = L - (L - H)$.

Let $H = H_1 - H_2$, and $L = H_1 + H_2$; let $\mathscr{L}$ be the Hilbert subspace of E with kernel L. We have $H_1 \leqq L$; so H_1 is the kernel of a Hilbert subspace $\mathscr{H}_1$ of $\mathscr{L}$ (Proposition 13). Let us use Proposition 9b. We have $H_1 = A_1 L$, where A_1 is a non-negative continuous Hermitian operator from $\mathscr{L}$ into itself; likewise

$H_2 = A_2 L$. So $H = AL$, with A continuous and Hermitian from $\mathscr{L}$ into $\mathscr{L}$. The spectral decomposition then gives a decomposition $A = B_1 - B_2$, where B_1 and B_2 are non-negative continuous alien Hermitian operators in $\mathscr{L}$. B_1 and B_2 define Hilbert subspaces in $\mathscr{L}$, $\mathscr{B}_1 = \sqrt{B_1}\mathscr{L}$ and $\mathscr{B}_2 = \sqrt{B_2}\mathscr{L}$, with intersection $\{0\}$, that is to say, they are alien (Proposition 16).

The kernels of $\mathscr{B}_1$ and $\mathscr{B}_2$ in E are $B_1 L$ and $B_2 L$, so they are alien; and $H = AL = B_1 L - B_2 L$ in $\mathscr{L}^+(E)$. □

Corollary 1. *For a Hermitian kernel H to be the difference of two non-negative kernels, it is necessary that the Hermitian form H which it defines on $E' \times E'$ (Equation (3.5)) is strongly continuous.*

Proof. It suffices to apply the Corollary of Proposition 10. □

Corollary 2. *For every Hermitian kernel H to be the difference of two non-negative kernels, it is necessary that every separately weakly continuous sesquilinear form on $E' \times E'$ is strongly continuous.*

Proof. It is indeed necessary, by Corollary 1, that, for every Hermitian kernel H, $\tilde{H}$ is strongly continuous; the same is also necessary for every kernel H, because it is of the form $A + iB$, where A and B are Hermitian kernels. But every separately weakly continuous sesquilinear form on $E' \times E'$ is $\tilde{H}$ of a kernel H. □

Example. Let $E = \mathscr{D}'$, the space of distributions on $\mathbb{R}^n$. Then $\bar{E}' = \mathscr{D}$. We know that there exist separately continuous bilinear forms on $\mathscr{D} \times \mathscr{D}$, which are not continuous ($\mathscr{D} \otimes_\varepsilon \mathscr{D}$ is different from $\mathscr{D} \otimes_\pi \mathscr{D}$, and does not have the same dual[(58)]). So there are Hermitian kernels relative to $\mathscr{D}'$ which are not differences of two non-negative kernels.

In situations where we use this Corollary, we will only have examples for which $\bar{E}'$ is not a Fréchet space (with respect to the strong topology); because if $\bar{E}'$ is a Fréchet space, every separately weakly continuous sesquilinear form on $E' \times E'$ is separately strongly continuous, and so strongly continuous.

However, there are Banach spaces E with Hermitian kernels that are not bounded from above by non-negative kernels. Indeed, let G be a reflexive Banach space that does not admit a Hilbert structure. Let $E = G \oplus \bar{G}'$. Then $\bar{E}' = \bar{G}' \oplus G$. There exists a natural Hermitian kernel H, the map $(\bar{x}', y) \mapsto (y, \bar{x}')$; it is indeed Hermitian, because $\langle (y, \bar{x}'), (x', \bar{y}) \rangle = 2\mathrm{Re}\langle x', y \rangle$ is real. If H was the difference of two non-negative kernels, there also exist non-negative alien kernels H_1 and H_2 such that $H = H_1 - H_2$, and we would have $E = H(\bar{E}') = (H_1 - H_2)(\bar{E}') \subset H_1(\bar{E}') + H_2(\bar{E}')$, such that the Hilbert subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ with kernels H_1 and H_2 would satisfy $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$ and $\mathscr{H}_1 + \mathscr{H}_2 = \mathscr{H}$ as sets. E would then admit a Hilbert structure, namely $\mathscr{H}_1 + \mathscr{H}_2$, hence the closed subspace G would also admit a Hilbert structure, which contradicts the hypothesis.

Proposition 38 leads us to introduce a stricter notion. A vector space $\mathscr{H}$ (over $\mathbb{C}$) will be called a Hermitian space, if it is equipped with a Hermitian form $(\cdot \mid \cdot)_{\mathscr{H}}$ with the following property: there exists a decomposition of $\mathscr{H}$ into a direct sum $\mathscr{H}_1 + \mathscr{H}_2$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are orthogonal, such that, with

(58) Schwartz [2], pages 115-116.

respect to the restriction of the Hermitian form, $\mathscr{H}_1$ is a Hilbert space, and with respect to the restriction of the negative of the Hermitian form, $\mathscr{H}_2$ is a Hilbert space.

We therefore have $\mathscr{H} = \mathscr{H}_1 - \mathscr{H}_2$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are alien Hilbert spaces. As we saw on page 86, we have $\gamma = \gamma_1 \oplus \gamma_2 \cdot \gamma(\mathscr{H}) = \gamma_1(\mathscr{H}_1) + \gamma_2(\mathscr{H}_2)$.

It can then obviously be equipped with the product topology $\mathscr{H}_1 \times \mathscr{H}_2$, with respect to which it is complete, and admits a Hilbert structure (namely, $\mathscr{H}_1 + \mathscr{H}_2$). The Hermitian form is continuous with respect to this topology.

The anti-dual of $\mathscr{H}$, with respect to this topology, is $\gamma(\mathscr{H}) \subset \bar{\mathscr{H}}^*$, and γ is an isomorphism from $\mathscr{H}$ onto $\bar{\mathscr{H}}'$. $\mathscr{H}_1$ and $\mathscr{H}_2$ are not given in the structure, we only assume their existence, and there are in general infinitely many possibilities for $\mathscr{H}_1$ and $\mathscr{H}_2$. On the other hand, the topology defined on $\mathscr{H}$ is intrinsic, because it is a Banach topology, and the anti-dual with respect to this topology is known, it is $\gamma(\mathscr{H})$. It is these Hermitian spaces which generalise best the Hilbert spaces.

A Hermitian subspace $\mathscr{H}$ of E is then a vector subspace equipped with the structure of a Hermitian space, such that the inclusion $\mathscr{H} \to E$ is continuous. It is then indeed an admissible pre-Hermitian subspace, because its Hermitian form is not degenerate, and $j^*(\bar{E}') \subset \bar{\mathscr{H}}'$ since j is continuous, and $\gamma(\mathscr{H}) = \bar{\mathscr{H}}'$. Moreover, $H(\bar{E}')$ is then dense in $\mathscr{H}$ since $j^*(\bar{E}')$ is dense in $\bar{\mathscr{H}}'$ (as j is injective), and $\theta = j^{-1}$ is an isomorphism from $\bar{\mathscr{H}}'$ onto $\mathscr{H}$. Proposition 38 then allows us to state:

Corollary 3. *The vector subspace of* $\mathrm{Herm}(E)$ *generated by the cone* $\mathrm{Hilb}(E)$ *is that of equivalence classes of Hermitian subspaces of* E*.*

Proof. If H is a difference of two non-negative kernels, there also exist non-negative alien kernels H_1 and H_2, such that $H = H_1 - H_2$. If $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert subspaces with kernels H_1 and H_2, H is the kernel of the Hermitian subspace $\mathscr{H}_1 - \mathscr{H}_2$. □

Remark. It could appear that Hermitian spaces are a good notion to introduce from the beginning. But:

1°) As we will see later (§13), two distinct Hermitian subspaces can have the same associated kernel. An equivalence relation is therefore again inevitable.

2°) Hermitian subspaces do not form a stable category with respect to the three operations: if $\mathscr{H}$ is a Hermitian subspace of E, and u a continuous linear map from E into F, $\ddot{u}(\mathscr{H})$ is not necessarily Hermitian. (It is only equivalent to some Hermitian subspaces, since its class is generated by some classes of Hilbert subspaces; but it can then be equivalent to infinitely many of them, without one of them having a particular reason to be associated to it. Naturally, if u is injective, $\ddot{u}(\mathscr{H}) = u(\mathscr{H})$ is Hermitian). Likewise, the sum $\mathscr{H}_1 \ddot{+} \mathscr{H}_2$ of two Hermitian subspaces is not necessarily Hermitian (but it is Hermitian if $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$).

§13. Unicity and multiplicity of Hermitian kernels

A Hermitian kernel $H \in \mathscr{L}^h(E)$, which is a difference of two non-negative kernels, is said to have unicity if there exists a unique Hermitian subspace of E

that admits it as its associated kernel; otherwise, it is said to have multiplicity. For example, a kernel $H \geqq 0$ has unicity, because the only Hermitian subspaces with kernel H are Hilbert subspaces (indeed, the square norm of an element of $H(\bar{E}')$ is non-negative if $H \geqq 0$, so, by continuity, the square norm of every element of $\mathscr{H}$ is non-negative) and uniqueness follows from Proposition 8. (Let us remark, on the other hand, that an admissible pre-Hermitian space which is not pre-Hilbert, i.e. a space with a form that is not non-negative, can have a non-negative kernel.) A Hermitian subspace is said to have unicity or multiplicity, depending on whether its associated kernel has unicity or multiplicity.

Proposition 39. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hermitian subspaces of E, with the same kernel H. If $\mathscr{H}_1 \subset \mathscr{H}_2$, these two Hermitian spaces coincide.*

Proof. Let us equip $\mathscr{H}_2$ with the topology canonically defined by its Hermitian structure; the inclusion $j : \mathscr{H}_2 \to E$ is continuous. If L_1 and L_2 are the kernels of $\mathscr{H}_1$ and $\mathscr{H}_2$ relative to $\mathscr{H}_2$, considered as maps $\bar{\mathscr{H}}_2' \to \mathscr{H}_2$, and if H is considered as a map $\bar{E}' \to \mathscr{H}_2$, we have $H = L_1$ and $j^* = L_2 j^*$. As $j^*(\bar{E}')$ is dense in $\bar{\mathscr{H}}_2'$ (since j is injective), we must have $L_1 = L_2$: $\mathscr{H}_1$ and $\mathscr{H}_2$ already have the same kernel L in $\mathscr{H}_2$. But then they both contain $L(\bar{\mathscr{H}}_2')$, and, on this subspace, have the same Hermitian form, defined by (4.3). As L, the kernel of $\mathscr{H}_2$, is the isomorphism $\gamma_2^{-1} : \bar{\mathscr{H}}_2' \to \mathscr{H}_2$, $L(\bar{\mathscr{H}}_2')$ is $\mathscr{H}_2$, and $\mathscr{H}_1$ and $\mathscr{H}_2$ are indeed the same vector space with the same Hermitian form. □

This Proposition shows that, when we don't have unicity, the diverse Hermitian subspaces with the same kernel H do not satisfy relations of inclusion. This already gives examples of non-Hilbert unicity: if one of the Hermitian subspaces with kernel H, as a set, is E itself, then we have unicity.

Proposition 40. *Let H be a Hermitian kernel, a difference $H_1 - H_2$ of two non-negative kernels, with the rank h of H_2 finite. Then, for every decomposition $H = K_1 - K_2$ into a difference of two non-negative alien kernels K_1 and K_2, the rank of K is a natural number $k \leqq h$. Moreover, the kernel H has unicity.*

Proof. Let $\mathscr{H}_1, \mathscr{H}_2, \mathscr{K}_1, \mathscr{K}_2$ be Hilbert subspaces with kernels H_1, H_2, K_1, K_2. $\mathscr{H}_2$ is of dimension h, since it has to be the closure of the subspace of finite dimension $H_2(\bar{E}')$, and so is equal to this subspace. We have $H_1 + K_2 = H_2 + K_1$; let us denote by L the kernel and $\mathscr{L}$ the corresponding Hilbert subspace. We have $\mathscr{H}_1 + \mathscr{K}_2 = \mathscr{H}_2 + \mathscr{K}_1 = \mathscr{L}$. So the codimension of $\mathscr{K}_1$ in $\mathscr{L}$ is at most h. But $\mathscr{K}_1$ and $\mathscr{K}_2$ are alien, and both are contained in $\mathscr{L}$, so $\dim(\mathscr{K}_2) \leqq h$, and as a consequence k, the rank of K_2, is at most h. The initial decomposition is now meaningless. So let us assume, to continue, that $H_1 - H_2$ and $K_1 - K_2$ are both decompositions into differences of two non-negative alien kernels. Then the proof above gives that the rank of K_2 is at most the rank of H_2, but also that the rank of H_2 is less than the rank of K_2, so these ranks are equal; let k be their common value. Then $\mathscr{H}_1$ and $\mathscr{K}_1$ both have codimension at most k in $\mathscr{L}$, since their sums with $\mathscr{K}_2$ and $\mathscr{H}_2$ give $\mathscr{L}$; it is also at least k, since they are alien to $\mathscr{H}_2$ and $\mathscr{K}_2$, so they have the same codimension k. But then $\mathscr{H}_1$ and $\mathscr{H}_2$ are alien and $\dim \mathscr{H}_2 = \operatorname{codim} \mathscr{H}_1$, so $\mathscr{H}_1 + \mathscr{H}_2 = \mathscr{L}$ and likewise $\mathscr{K}_1 + \mathscr{K}_2 = \mathscr{L}$. The Hermitian spaces have the same kernel H, and $\mathscr{H}_1 - \mathscr{H}_2$ and $\mathscr{K}_1 - \mathscr{K}_2$ coincide as sets with $\mathscr{L}$, and hence they are the same Hermitian space by Proposition 39. As every Hermitian space $\mathscr{H}$ with kernel H is a difference of two alien Hilbert spaces, the kernel H indeed has unicity. □

Thus the only kernels that can possibly have multiplicity are those that can be written as $H_1 - H_2$, where H_1 and H_2 are alien and non-negative, both with infinite ranks. But the example which follows Proposition 39 shows that, even in this case, H can have unicity.

Let us give an example of a kernel that has multiplicity. Let E be a Hilbert space, and let $\mathscr{A}$ and $\mathscr{B}$ be two closed vector subspaces, such that $\mathscr{A} \cap \mathscr{B} = \{0\}$, and such that $\mathscr{A} + \mathscr{B}$ is dense in E without being equal to E. Then their orthogonal complements $\mathscr{A}^\times$ and $\mathscr{B}^\times$ in E with respect to the Hilbert structure of E have the same property. Let us equip all of them with the Hilbert structure induced by E. Let us identify $\bar{E}'$ with E via the Hilbert structure. Let us denote by A and B the kernels of $\mathscr{A}$ and $\mathscr{B}$ in E; those of $\mathscr{A}^\times$ and $\mathscr{B}^\times$ are $I - A$ and $I - B$, since $\mathscr{A} + \mathscr{A}^\times = E$. We thus have $A - B = (I - B) - (I - A)$. However, the Hermitian subspaces $\mathscr{A} - \mathscr{B}$ and $\mathscr{B}^\times - \mathscr{A}^\times$ do not coincide; indeed, the corresponding sets are $\mathscr{A} + \mathscr{B}$ and $\mathscr{A}^\times + \mathscr{B}^\times$; if they coincide, it would be a vector subspace containing $\mathscr{A}$ and $\mathscr{A}^\times$, so E, but $\mathscr{A} + \mathscr{B} \neq E$. $A - B$ is thus a kernel with multiplicity relative to E, and $\mathscr{A} - \mathscr{B}$ is a Hermitian subspace with multiplicity. Hence the multiplicity can arise even when E is a Hilbert space. It is well known that the above situation requires that $\mathscr{A}$ and $\mathscr{B}$ are of infinite dimensions. The result would be the same if we simply assumed $\mathscr{A} \cap \mathscr{B} = \{0\}$, and $\mathscr{A} + \mathscr{B}$ not closed. If, indeed, E_1 is then the closure of $\mathscr{A} + \mathscr{B}$ in E, $\mathscr{A} - \mathscr{B}$ has multiplicity in E_1, and so in E. It is well known that, if $\mathscr{A}$ and $\mathscr{B}$ are two closed alien vector subspaces of a Hilbert space, a necessary and sufficient condition for $\mathscr{A} + \mathscr{B}$ not to be closed is[(59)]:

$$\text{(T)} \qquad \begin{cases} \textit{for any } \varepsilon > 0, \textit{ there exist } a \in \mathscr{A} \text{ and } b \in \mathscr{B}, \textit{ with norm 1, such} \\ \textit{that } \|a - b\| \leqq \varepsilon; \textit{ or such that } (a \mid b) \geqq 1 - \varepsilon. \end{cases}$$

When this condition (T) is satisfied, we say that $\mathscr{A}$ and $\mathscr{B}$ are in position (T) (tangent). So if $\mathscr{A}$ and $\mathscr{B}$ are two alien closed vector subspaces of a Hilbert space E in position (T), $\mathscr{A} - \mathscr{B}$ has multiplicity in E.

Let us now move onto a more general situation whereby everything is transformed by a continuous linear map. Let $\mathscr{A}$ and $\mathscr{B}$ be two Hilbert subspaces of any E. Let us assume that we can find a structure of a pre-Hilbert subspace $\mathscr{H}$ of E on the set $\mathscr{A} + \mathscr{B}$, inducing on $\mathscr{A}$ and $\mathscr{B}$ their Hilbert structures, but in which they are in position (T). $\mathscr{H}$ is not complete, from above. Let us then show that the Hermitian subspace $\mathscr{A} - \mathscr{B}$ of E has multiplicity. Indeed, let $\hat{\mathscr{H}}$ be the completion of $\mathscr{H}$; the inclusion j of $\mathscr{H}$ in E extends to a continuous linear map $\hat{\jmath}$ from $\hat{\mathscr{H}}$ into E, which will not in general be injective. Let $\mathscr{N}$ be the kernel of $\hat{\jmath}$, and $\mathscr{K}$ the orthogonal complement of $\mathscr{N}$ in $\hat{\mathscr{H}}$. Let p be the orthogonal projection of $\hat{\mathscr{H}}$ onto $\mathscr{K}$. The image $\hat{\jmath}(\hat{\mathscr{H}})$ is the Q-completion $\hat{\mathscr{H}}_Q$ of $\mathscr{H}$ in E (Proposition 1b). The subspaces $\mathscr{A}$ and $\mathscr{B}$ are exactly in the situation envisaged previously, with respect to $\hat{\mathscr{H}}$: closed, equipped with the induced Hilbert structure, $\mathscr{A} \cap \mathscr{B} = \{0\}$, $\mathscr{A} + \mathscr{B}$ dense in but distinct from the whole space $\hat{\mathscr{H}}$, since $\mathscr{A}$ and $\mathscr{B}$ are in position (T) in $\mathscr{H}$, and so in $\hat{\mathscr{H}}$. Since j is the identity on $\mathscr{H} = \mathscr{A} + \mathscr{B}$, we have $(\mathscr{A} + \mathscr{B}) \cap \mathscr{N} = \{0\}$, and

(59) Saying that (T) is satisfied is indeed the same as saying that the map $(a, b) \mapsto (a, b)$ from $\mathscr{A} + \mathscr{B}$ into $\mathscr{A} \times \mathscr{B}$ is discontinuous. But this is the same as saying that $\mathscr{A}$ and $\mathscr{B}$ are not topologically complementary in $\mathscr{A} + \mathscr{B}$; according to Banach's Theorem (Bourbaki [1], Chapter I, §3, n°3, Corollary 4 of Theorem 1), this is equivalent to saying that $\mathscr{A} + \mathscr{B}$ is not complete, or not closed in E.

p is injective on $\mathscr{A} + \mathscr{B}$. Let us consider alien subspaces $p(\mathscr{A})$ and $p(\mathscr{B})$ of $\mathscr{K}$, equipped with the transported Hilbert structures; we have $\hat{\jmath}p(\mathscr{A}) = \mathscr{A}$ and $\hat{\jmath}p(\mathscr{B}) = \mathscr{B}$. The Hilbert subspace "differences" $\mathscr{K} - p(\mathscr{A})$ and $\mathscr{K} - p(\mathscr{B})$ (in the sense of Proposition 14) are again alien in $\mathscr{K}$. Indeed, if they were not alien, there would exist a Hilbert subspace $\mathscr{L}$ of $\mathscr{K}$, distinct from $\{0\}$, such that $\mathscr{L} \leqq \mathscr{K} - p(\mathscr{A})$ and $\mathscr{L} \leqq \mathscr{K} - p(\mathscr{B})$; we would then also have $\mathscr{L} + \mathscr{N} \leqq (\mathscr{K} - p(\mathscr{A})) + \mathscr{N} = (\mathscr{K} + \mathscr{N}) - p(\mathscr{A}) = \hat{\mathscr{H}} - p(\mathscr{A})$ (the two Hilbert subspaces of $\hat{\mathscr{H}}$ on either side of the first equality sign have the same associated kernel, and so are indeed equal), and likewise $\mathscr{L} + \mathscr{N} \leqq \hat{\mathscr{H}} - p(\mathscr{B})$; thence $\mathscr{L} \leqq \hat{\mathscr{H}} - (p(\mathscr{A}) + \mathscr{N}) \leqq \hat{\mathscr{H}} - \mathscr{A}$ and $\mathscr{L} \leqq \hat{\mathscr{H}} - \mathscr{B}$, which is impossible, since $\hat{\mathscr{H}} - \mathscr{A} = \mathscr{A}^{\times}$ and $\mathscr{H} - \mathscr{B} = \mathscr{B}^{\times}$ are alien in $\hat{\mathscr{H}}$, as $\mathscr{A} + \mathscr{B}$ is dense. We can thus form the Hermitian space $(\mathscr{K} - p(\mathscr{B})) - (\mathscr{K} - p(\mathscr{A}))$ in $\mathscr{K}$; it has the same kernel as $p(\mathscr{A}) - p(\mathscr{B})$ relative to $\mathscr{K}$; let us show that these two Hermitian subspaces of $\mathscr{K}$ are always distinct. They indeed can only coincide if there exists a Hermitian form on $\mathscr{K}$ which is non-negative on $p(\mathscr{A})$ and non-positive on $\mathscr{K} - p(\mathscr{A})$; we then have $p(\mathscr{A}) \cap (\mathscr{K} - p(\mathscr{A})) = \{0\}$; but then $\mathscr{K} = p(\mathscr{A}) + (\mathscr{K} - p(\mathscr{A}))$ shows that $p(\mathscr{A})$ and $\mathscr{K} - p(\mathscr{A})$ are two closed orthogonal subspaces of $\mathscr{K}$, with the induced Hilbert structure. As $\mathscr{A}$ has the Hilbert structure induced by $\hat{\mathscr{H}}$ and as p strictly reduces the norms of elements of $\mathscr{K}^c$, this means that $\mathscr{A} \subset \mathscr{K}$; likewise $\mathscr{B} \subset \mathscr{K}$. But then, as $\mathscr{A} + \mathscr{B}$ is dense in $\hat{\mathscr{H}}$, we have $\mathscr{K} = \hat{\mathscr{H}}$ and $\mathscr{N} = \{0\}$; but then we know that $\mathscr{A} - \mathscr{B}$ and $(\hat{\mathscr{H}} - \mathscr{B}) - (\hat{\mathscr{H}} - \mathscr{A})$ are distinct, since we have $\mathscr{A} \cap \mathscr{B} = \{0\}$ and $\mathscr{A} + \mathscr{B}$ is dense in but distinct from $\hat{\mathscr{H}}$. This indeed shows that $p(\mathscr{A}) - p(\mathscr{B})$ and $(\mathscr{K} - p(\mathscr{B})) - (\mathscr{K} - p(\mathscr{A}))$ are distinct Hermitian subspaces of $\mathscr{K}$, with the same kernel in $\mathscr{K}$. Since, then, $\hat{\jmath}$ is injective on $\mathscr{K}$, their images under $\hat{\jmath}$ have the same properties in E; but the image of the first is $\mathscr{A} - \mathscr{B}$, so $\mathscr{A} - \mathscr{B}$ has multiplicity in E, as we had stated.

This will allow us to give large classes of kernels with multiplicity.

Let H and K be two non-negative kernels, with $K \geqq H$; we will say that H is K-compact if, denoting the corresponding Hilbert subspaces by $\mathscr{H}$ and $\mathscr{K}$, the inclusion $\mathscr{H} \to \mathscr{K}$ is compact; or if the unit ball of $\mathscr{H}$ is relatively compact in $\mathscr{K}$ (in which case it is compact, since it is weakly compact).

Proposition 41. *Let H_1 and H_2 be two non-negative alien kernels of E, both with infinite rank. Suppose that there exists a kernel $K \geqq 0$ such that H_1 and H_2 are K-compact; then the Hermitian kernel $H_1 - H_2$ has multiplicity.*

Proof. Let $\mathscr{K}$, $\mathscr{H}_1$ and $\mathscr{H}_2$ be the Hilbert subspaces of E associated with the above kernels. Let i_1 be the inclusion of $\mathscr{H}_1$ in $\mathscr{K}$, which is compact; its adjoint i_1^* is thus also compact from $\mathscr{K}$ into $\mathscr{H}_1$, identified with $\bar{\mathscr{K}}'$ and $\bar{\mathscr{H}}_1'$; a fortiori the kernel $A_1 = i_1 i_1^* : \mathscr{K} \to \mathscr{K}$ of $\mathscr{H}_1$ relative to $\mathscr{K}$ is compact. But A_1 is also non-negative and Hermitian; let us use its spectral decomposition, which is discrete. We can find an orthonormal basis $(u_i)_{i \in I}$, with I containing the set $\mathbf{N}$ of the natural numbers, such that $Au_n = \varepsilon_n u_n$ for $n \in \mathbf{N}$, $Au_i = 0$ for $i \notin \mathbf{N}$, with the sequence of the $\varepsilon_n > 0$ converging to 0 as $n \to \infty$ (it is a classical result that these values, each counted with its order of multiplicity, form a finite sequence, or an infinite sequence converging to 0; the finite case is excluded since H_1 is assumed to have infinite rank). We moreover have $\mathscr{H}_1 = \sqrt{A_1}\mathscr{K}$ (Corollary 5 of Proposition 21), such that $a_n = \sqrt{\varepsilon_n} u_n$ form an orthonormal basis of $\mathscr{H}_1$. We can likewise find an orthonormal basis $(v_i)_{i \in I}$ of $\mathscr{K}$, and a sequence $\eta_n > 0$ converging to 0 as $n \to \infty$, such that $b_n = \sqrt{\eta_n} v_n$ form an orthonormal basis of $\mathscr{H}_2$.

Let us now construct a pre-Hilbert structure $\mathscr{H}$ on $\mathscr{H}_1 + \mathscr{H}_2$ as follows. It will preserve the orthonormality of a_n, as well as that of b_n; a_m and b_n will be orthogonal if $m \neq n$; and we let $(a_n \mid b_n) = 1 - \delta_n$, $0 < \delta_n < 1$. Put differently, for every point in $\mathscr{H}_1 + \mathscr{H}_2$ of the form

$$\sum_{n=0}^{\infty}(\alpha_n a_n + \beta_n b_n), \qquad \text{with} \qquad \sum_{n=0}^{\infty}|\alpha_n|^2 < +\infty, \sum_{n=0}^{\infty}|\beta_n|^2 < +\infty,$$

we will let

$$\left\| \sum_{n=0}^{\infty}(\alpha_n a_n + \beta_n b_n) \right\|_{\mathscr{H}}^2 = \sum_{n=0}^{\infty} \left[|\alpha_n|^2 + |\beta_n|^2 + 2\mathrm{Re}(1-\delta_n)\alpha_n \bar{\beta}_n \right]. \tag{13.1}$$

(The series on the right-hand side is indeed always convergent, and is indeed non-negative by dint of the hypothesis $0 < \delta_n < 1$. Hence we do have a Hausdorff pre-Hilbert structure on $\mathscr{H}$). This structure induces on $\mathscr{H}_1$ and $\mathscr{H}_2$ their initial structures; moreover, if δ_n converge to 0 as $n \to \infty$, $\mathscr{H}_1$ and $\mathscr{H}_2$ are in position (T) in $\mathscr{H}$, since $(a_n \mid b_n)_{\mathscr{H}} = 1 - \delta_n$. It remains to show that we can choose the sequence δ_n such that $\mathscr{H}$ is a pre-Hilbert subspace of E, i.e. such that its unit ball B is bounded in E; then what came just before the statement of the present Proposition will indeed show that the Hermitian subspace $\mathscr{H}_1 - \mathscr{H}_2$ of E has multiplicity, which is the result we were after. It suffices to show that B is weakly bounded. So let e' be in E'. We have, for $h = \sum_{n=0}^{\infty}(\alpha_n a_n + \beta_n b_n) \in \mathscr{H}$:

$$\langle h, e' \rangle = \sum_{n=0}^{\infty}(\alpha_n \langle a_n, e' \rangle + \beta_n \langle b_n, e' \rangle). \tag{13.2}$$

Let m be a natural number such that $\varepsilon_n < 1$ and $\eta_n < 1$ for $n > m$.

The sum $\sum\limits_{n \leqq m}$ immediately admits the upper bound

$$|\sum_{n \leqq m}(\alpha_n \langle a_n, e' \rangle + \beta_n \langle b_n, e' \rangle)| \leqq A_{e'} \| \sum_{n \leqq m}(\alpha_n a_n + \beta_n b_n) \|_{\mathscr{H}} \tag{13.2b}$$

because every linear form on a vector space of finite dimension is bounded above by the norm, up to a factor. In addition,

$$\| \sum_{n \leqq m} \|_{\mathscr{H}} \leqq \| \sum_{n=0}^{\infty} \|_{\mathscr{H}} = \|h\|_{\mathscr{H}},$$

thence

$$|\sum_{n \leqq m}(\alpha_n \langle a_n, e' \rangle + \beta_n \langle b_n, e' \rangle)| \leqq A_{e'} \|h\|_{\mathscr{H}}. \tag{13.2c}$$

Consider now $\sum_{n>m}$:

$$\sum_m = |\sum_{n>m}(\alpha_n \langle a_n, e' \rangle + \beta_n \langle b_n, e' \rangle)|$$

$$= \sum_{n>m} (\alpha_n \sqrt{\varepsilon_n} \langle u_n, e' \rangle + \beta_n \sqrt{\eta_n} \langle v_n, e' \rangle)|$$

$$\leqq \left(\sum_{n>m} \varepsilon_n |\alpha_n|^2 \right)^{1/2} \left(\sum_{n>m} |\langle u_n, e' \rangle|^2 \right)^{1/2}$$

$$+ \left(\sum_{n>m} \eta_n |\beta_n|^2 \right) \left(\sum_{n>m} |\langle v_n, e' \rangle|^2 \right)^{1/2}, \tag{13.3}$$

by applying the Cauchy-Schwarz inequality here.

As the u_n and the v_n are subsets of Hilbert bases of $\mathscr{K}$, we have, by Corollary 5 of Proposition 19,

$$\left(\sum_{n>m}^{\infty} |\langle u_n, e' \rangle|^2 \right)^{1/2} \leqq B_{e'}, \qquad \left(\sum_{n>m}^{\infty} |\langle v_n, e' \rangle|^2 \right)^{1/2} \leqq B_{e'}, \tag{13.4}$$

where $0 \leqq B_{e'} < +\infty$ (where $B_{e'}$ is dependent on e'). Then

$$\sum_m \leqq B_{e'} \left[\left(\sum_{n>m}^{\infty} \varepsilon_n |\alpha_n|^2 \right)^{1/2} + \left(\sum_{n>m}^{\infty} \eta_n |\beta_n|^2 \right)^{1/2} \right]$$

$$\leqq 2B_{e'} \left[\sum_{n>m}^{\infty} (\varepsilon_n |\alpha_n|^2 + \eta_n |\beta_n|^2) \right]^{1/2}.$$

Let us suppose that we can choose δ_n such that, for $n > m$:

$$\varepsilon_n |\alpha_n|^2 + \eta_n |\beta_n|^2 \leqq |\alpha_n|^2 + |\beta_n|^2 + 2(1 - \delta_n) \mathrm{Re}(\alpha_n \bar{\beta}_n). \tag{13.6}$$

We will then have

$$|\langle h, e' \rangle| \leqq C_{e'} \|h\|_{\mathscr{H}}, \qquad C_{e'} = A_{e'} + 2B_{e'}, \tag{13.7}$$

and the unit ball B of $\mathscr{H}$ will indeed be weakly bounded, which will finish the proof. But, (13.6) is equivalent to

$$(1 - \varepsilon_n)(\alpha_n)^2 + (1 - \eta_n)|\beta_n|^2 + 2(1 - \delta_n) \mathrm{Re}(\alpha_n \bar{\beta}_n) \geqq 0. \tag{13.8}$$

As $1 - \varepsilon_n > 0$ and $1 - \eta_n > 0$ for $n > m$, (13.8) is equivalent to:

$$(1 - \varepsilon_n)(1 - \eta_n) - (1 - \delta_n)^2 > 0. \tag{13.9}$$

It will thus suffice to choose δ_n arbitrary for $n \leqq m$ ($0 < \delta_n < 1$), and for $n > m$:

$$1 - \delta_n = \sqrt{(1 - \varepsilon_n)(1 - \eta_n)} \left(1 - \frac{1}{n} \right). \tag{13.10}$$

We then indeed have $0 < 1 - \delta_n < 1$, so $0 < \delta_n < 1$, and

$$1 - \delta_n \to 1 \qquad \text{as } n \to \infty, \qquad \text{so } \delta_n \to 0.$$

□

Corollary 1. *Let E be a locally convex, quasi-complete Hausdorff topological vector space, with the following property: for every Hilbert subspace $\mathscr{H}$ of E, there exists another Hilbert subspace $\mathscr{K}$ such that the inclusion of $\mathscr{H}$ in $\mathscr{K}$ is compact. Then, a Hermitian kernel of E that is a difference $H_1 - H_2$ of two non-negative alien kernels, has unicity if and only if H_1 and H_2 are of finite rank.*

This follows trivially from Propositions 40 and 41.

Corollary 2. *Let E be the dual of a barrelled nuclear space. A Hermitian kernel H of E that is a difference $H_1 - H_2$ of two non-negative alien kernels, has unicity if and only if H_1 and H_2 are of finite rank.*

Proof. We have $E = F'$, where F is barrelled and nuclear, and so reflexive; $\bar{E}' = F$. Let $\mathscr{H}$ be a Hilbert subspace of E. By taking the adjoint of the inclusion $\mathscr{H} \to E$, we have a continuous linear map $\bar{E}' \to \mathscr{H}$ (where $\mathscr{H}$ is identified with $\bar{\mathscr{H}}'$). But we know that every continuous linear map from a nuclear space $F = \bar{E}'$ into a Banach space factorises into a product of nuclear operators. There exist Banach spaces $\mathscr{A}$ and $\mathscr{B}$ such that $\bar{E}' \to \mathscr{H}$ factorises into $\bar{E}' \to \mathscr{A} \to \mathscr{B} \to \mathscr{H}$, with all these maps being nuclear. But every nuclear map from a Banach space into another factorises into a product of continuous maps, with a Hilbert intermediate space; so there exists a factorisation of $\mathscr{A} \to \mathscr{B}$ into $\mathscr{A} \to \mathscr{K}_1 \to \mathscr{B}$, where the maps are continuous and linear, and $\mathscr{K}_1$ is a Hilbert space. We thus have $\bar{E}' \to \mathscr{K}_1 \to \mathscr{H}$, where $\mathscr{K}_1 \to \mathscr{H}$ is nuclear and so compact. By taking the adjoint, the inclusion $\mathscr{H} \to E$ factorises into $\mathscr{H} \to \mathscr{K}_1 \to E$, where $\mathscr{H} \to \mathscr{K}_1$ is compact. If $\mathscr{H}$ is the image of $\mathscr{K}_1$ in E, with the image Hilbert structure, we have the factorisation $\mathscr{H} \to \mathscr{K}_1 \to \mathscr{K} \to E$, so $\mathscr{H} \to \mathscr{K} \to E$, where $\mathscr{H} \to \mathscr{K}$ is compact; $\mathscr{K} \to E$ and $\mathscr{H} \to E$ are inclusions, and so is $\mathscr{H} \to \mathscr{K}$; put differently, $\mathscr{K}$ is a Hilbert subspace of E and $\mathscr{H} \subset \mathscr{K}$, such that the inclusion $\mathscr{H} \to \mathscr{K}$ is compact. It then suffices to apply Corollary 1. □

Example. $E = \mathscr{D}, \mathscr{D}', \mathscr{E}, \mathscr{E}', \mathscr{S}, \mathscr{S}', \mathscr{O}_M, \mathscr{O}'_M, \mathscr{C}, \mathscr{O}'_C$ [60].

Let us now suppose that $\bar{E}'$ is a subspace of E, in the setting of §11. Let $\mathscr{H}$ be a normal Hermitian subspace of E. Its anti-dual $\bar{\mathscr{H}}'$ is then a normal Hermitian subspace of E, if we transport the Hermitian structure of $\mathscr{H}$ onto $\bar{\mathscr{H}}'$ via the isomorphism γ. Proposition 28b then remains valid. On the other hand, there does not seem to be an equivalent of Propositions 29 and 30.

If the kernel H of $\mathscr{H}$ is extendable, it is necessary, for $\mathscr{H}$ to be normal, that H has a Hermitian inverse, since the kernel $\bar{H}'$ of $\bar{\mathscr{H}}'$ is Hermitian; but this has no reason to be sufficient. (If, by the way, H has some bilateral inverse L, it also has a Hermitian inverse $(L + L^*)/2$). It would be interesting to extend potential theory and the Dirichlet problem to Hermitian differential operators as they were presented in §11.

Index of Definitions and Notations

[60] All these spaces are barrelled, nuclear and reflexive, hence so are their duals. See, for example, Schwartz [1], Chapters 3 and 7, and Grothendieck [1], Chapter II, §2, n°3, page 54.

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Institut Henri Poincaré
Faculté des Sciences
Université de Paris.